

TRUNCATED ABELIAN LATTICE-ORDERED GROUPS II: THE POINTFREE (MADDEN) REPRESENTATION

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ABSTRACT. This is the second of three articles on the topic of truncation as an operation on divisible abelian lattice-ordered groups, or simply ℓ -groups. This article uses the notation and terminology of the first article and assumes its results. In particular, we refer to an ℓ -group with truncation as a truncated ℓ -group, or simply a trunc, and denote the category of trunks with truncation morphisms by **AT**.

Here we develop the analog for **AT** of Madden's pointfree representation for **W**, the category of archimedean ℓ -groups with designated order unit. More explicitly, for every archimedean trunc A there is a regular Lindelöf frame L equipped with a designated point $*$: $L \rightarrow 2$, a subtrunc \hat{A} of $\mathcal{R}_0 L$, the trunc of pointed frame maps $\mathcal{O}_0 \mathbb{R} \rightarrow L$, and a trunc isomorphism $A \rightarrow \hat{A}$. A pointed frame map is just a frame map between frames which commutes with their designated points, and $\mathcal{O}_0 \mathbb{R}$ stands for the pointed frame which is the topology $\mathcal{O} \mathbb{R}$ of the real numbers equipped with the frame map of the insertion $0 \rightarrow \mathbb{R}$. $(L, *)$ is unique up to pointed frame isomorphism with respect to its properties. Finally, we reprove an important result from the first article, namely that **W** is a non-full monoreflective subcategory of **AT**.

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1. INTRODUCTION

1.1. A brief synopsis. We develop the analog for trunks of Madden's pointfree representation for \mathbf{W} , the category of archimedean ℓ -groups with order unit ([6], [7], [2]). We begin by showing in Section 2 that, for an arbitrary trunk A , the truncation kernels form a regular Lindelöf frame $\mathcal{K}A$. In Section 3 we then provide a subtrunk \underline{A} of $\mathcal{K}A$ and a trunk isomorphism $\kappa_A : A \rightarrow \underline{A}$. This much directly parallels Madden's development in \mathbf{W} . But here an obstacle rears its head, an obstacle which is invisible in \mathbf{W} .

Although the representation $\kappa_A : A \rightarrow \underline{A} \leq \mathcal{K}A$ is faithful, i.e., one-one, it is not functorial, as we show by example. However, we restore functoriality by the simple stratagem of attaching a designated point to each frame under consideration and restricting our attention to the frame maps which respect the points by commuting with them. This requires a systematic study of the category of pointed frames, and we carry this out in Section 4. The corresponding algebraic construct is \mathcal{R}_0L , the trunc of pointed frame maps $\mathcal{O}_0\mathbb{R} \rightarrow L$, where $\mathcal{O}_0\mathbb{R}$ designates the pointed frame of the real numbers, the point being the frame map of the insertion $0 \rightarrow \mathbb{R}$. (For instance, if L is compact and regular then $L = \mathcal{O}X$ for some compact Hausdorff space X with designated point x_0 , and \mathcal{R}_0L is isomorphic to \mathcal{D}_0X , the trunc of continuous almost-finite extended-real valued functions on X which vanish at x_0 .) The section culminates in the development of the spectrum $\mathcal{M}A$ of the trunk A ; it is the pointed frame obtained by formally adjoining a point to the frame $\mathcal{K}A$ of truncation kernels.

All that has gone before is preparation for Section 5. There we work out the desired representation $A \rightarrow \hat{A} \leq \mathcal{R}_0\mathcal{M}A$, and show it to be faithful and functorial. We conclude

the paper with Section 6, in which we show that the unital objects form a non-full monoreflexive subcategory of the category of trunks, thereby confirming the proof of the same fact in [1].

For background on rings of continuous functions, we direct the reader to Gillman and Jerison's masterpiece [4], and for a general reference on ℓ -groups, to Darnel's fine text [3].

1.2. The basic definitions. For the purposes of this article, we define truncation as follows.

1.2.1. Definition. A *truncation* on an ℓ -group A is a unary operation $A^+ \rightarrow A^+$, written $a \mapsto \bar{a}$, which has the following properties for all $a, b \in A^+$.

- ($\mathfrak{T}1$) $a \wedge \bar{b} \leq \bar{a} \leq a$
- ($\mathfrak{T}2$) If $\bar{a} = 0$ then $a = 0$.
- ($\mathfrak{T}3$) If $na = \overline{na}$ for all n then $a = 0$.
- ($\mathfrak{T}4$) $\bigcap_{\mathbb{N}} a \ominus n^{\perp\perp} = 0$ for all $a \in A^+$. The symbol $a \ominus n$ stands for $n((a/n) \ominus 1) = a - na/n$. This axiom can therefore be formulated as
- ($\mathfrak{T}4'$) $\forall b > 0 \exists c \exists n (0 < c \leq b \text{ and } c \wedge a \ominus n = 0)$.

An ℓ -group equipped with a truncation is called a *truncated ℓ -group*, or *trunc* for short. A *truncation morphism* is an ℓ -homomorphism $f : A \rightarrow B$ between trunks which preserves the truncation, i.e., $f(\bar{a}) = \overline{f(a)}$ for all $a \in A^+$. We denote the category of trunks with truncation morphisms by **AT**.

The definition of truncation in [1] requires only the first three axioms; the fourth appears in that article as the requirement that each element of A^+ be archimedean, and it follows from [1, 5.3.1] that A itself is archimedean. In summary, we get the following.

1.2.2. Proposition. *The following are equivalent for an ℓ -group A .*

- (1) *A is truncated in the sense of Definition 1.2.1.*
- (2) *A is archimedean and truncated in the sense of [1].*
- (3) *A is archimedean and satisfies ($\mathfrak{T}1$) and ($\mathfrak{T}2$).*

Proof. The discussion prior to the proposition establishes the equivalence of the first two conditions. The equivalence of the second and third is just the observation that an archimedean trunc has no infinitesimals and therefore satisfies ($\mathfrak{T}3$). [1, 5.3.1]. \square

We reiterate two points for emphasis.

- *The trunks considered in this article are trunks in the sense of [1]. As a result, all the descriptive results from [1], including the identities in Section 3.3, apply to the more specialized trunks considered here.*
- *The trunks considered in this article are necessarily archimedean.*

1.2.3. Definition. A trunc A is said to be *unital with unit u* if $\bar{a} = a \wedge u$ for all $a \in A^+$. We denote by **W** the full subcategory of **AT** comprised of the unital trunks.

2. TRUNCATION IDEALS

We have set before us the task of developing a representation theory for \mathbf{AT} generalizing the pointfree Madden representation for \mathbf{W} . The universal objects for the latter are of the form $\mathcal{R}M$, the \mathbf{W} -object of frame maps $\mathcal{O}\mathbb{R} \rightarrow M$ for some frame M . In fact, an arbitrary \mathbf{W} -object A is captured as a subobject of $\mathcal{R}MA$, where $\mathcal{M}A$, the *Madden frame of A* , is the frame of \mathbf{W} -kernels of A . It is therefore to be expected that the frame of kernels of the morphisms of \mathbf{AT} plays a central role in the representation of trunks. We will refer to such kernels as *truncation kernels*. Our first task is to develop criteria which will enable us to recognize them.

2.1. An internal characterization. An *archimedean kernel* of an ℓ -group A is the set of elements sent to zero by an ℓ -homomorphism into an archimedean codomain. Of the several known characterizations of such kernels, we will use only the simplest, which we reprove here in the interests of a self-contained treatment. We continue to assume that A represents an arbitrary trunc.

2.1.1. Lemma. *A subset K of a trunc A is an archimedean kernel iff it is a convex ℓ -subgroup such that, for all $a \in A^+$,*

$$(\exists c \in A^+ \forall n \in \mathbb{N} (na - c)^+ \in K) \implies a \in K.$$

Proof. The quotient group A/K can be endowed with an order making the quotient map an ℓ -homomorphism iff K is a convex ℓ -subgroup. We must show that the archimedean property of A/K is equivalent to the condition displayed above. This follows directly from the fact that, for $a, c \in A^+$ and $n \in \mathbb{N}$,

$$\begin{aligned} (na - c)^+ \in K &\iff K = K + (na - c) \vee 0 \iff K + c = (K + na) \vee (K + c) \iff \\ &K + na \leq K + c \iff n(K + a) \leq K + c. \end{aligned} \quad \square$$

We shall say that a subset K of a trunc A is *absorbing* if $\overline{a} \in K$ implies $a \in K$ for all $a \in A^+$.

2.1.2. Lemma. *A subset $K \subseteq A$ is a truncation kernel iff it is an absorbing archimedean kernel.*

Proof. We have already mentioned that trunks are archimedean, hence a truncation kernel must be an archimedean kernel. And the absorbing property of a truncation kernel K is clearly necessitated by the fact that A/K must satisfy truncation axiom $(\mathfrak{T}2)$. On the other hand, if K is an absorbing archimedean kernel then define

$$\overline{K + a} = K + \overline{a}, \quad a \in A^+.$$

This is the only definition of truncation on A/K which makes the quotient map into a truncation morphism. Moreover, the truncation is well-defined just because $|\overline{a} - \overline{b}| \leq |a - b|$, $a, b \in A^+$, by [1, Lemma 3.3.1(4)]. It is then straightforward to verify that the operation thus defined satisfies the truncation axioms. \square

2.1.3. Corollary. *The truncation kernels of a \mathbf{W} -object coincide with its \mathbf{W} -kernels.*

Proof. A **W**-kernel is just an archimedean kernel K such that $a \wedge 1 \in K$ forces $a \in K$ for all $a \in A^+$. But the truncation in a **W**-object is taken to be $\bar{a} = a \wedge 1$, so the latter condition is exactly the absorbing property of K . \square

2.2. An external characterization. We use $\mathcal{K}A$ to designate the family of truncation ideals of A , considered as a lattice in the inclusion order. Our task for the remainder of the section is to show that $\mathcal{K}A$ is a regular Lindelöf frame; such a frame is completely regular. This we do in Theorem 2.3.5.

First, observe that $\mathcal{K}A$ is closed under intersection, a consequence of the fact that **AT** is evidently closed under products. This leads to the notion of the truncation kernel generated by a subset $B \subseteq A$, which we denote

$$[B] \equiv \bigcap \{K : \mathcal{K}A \ni K \supseteq B\}.$$

It is helpful to have an external description of $[B]$, and such a description requires a little terminology. Every ordinal number α can be expressed in the form $\alpha = \beta + k$ for a unique finite ordinal k and limit ordinal β . (We take $\beta = 0$ to be a limit ordinal.). We will say that α is *even* or *odd* depending on whether k is even or odd.

For a subset $B \subseteq A$, let $\langle B \rangle$ designate the convex ℓ -subgroup generated by B . Now define

$$\begin{aligned} B^0 &\equiv \langle B \rangle, \\ B^{\alpha+1} &\equiv \langle a \in A^+ : \exists c \in A^+ \forall n \in \mathbb{N} (n|a| - c)^+ \in B^\alpha \rangle, \alpha \text{ even}, \\ B^{\alpha+1} &\equiv \langle a \in A^+ : \overline{|a|} \in B^\alpha \rangle, \alpha \text{ odd}, \\ B^\beta &\equiv \bigcup_{\alpha < \beta} B^\alpha, \beta \text{ a limit ordinal}, \\ B^\infty &\equiv B^\alpha \text{ for some (any) } \alpha \text{ such that } B^\alpha = B^{\alpha+1}. \end{aligned}$$

2.2.1. Lemma. $[B] = B^\infty = B^{\omega_1}$ for any subset $B \subseteq A$.

Proof. This follows directly from Lemma 2.1.2. \square

Polars are good examples of truncation kernels.

2.2.2. Corollary. For any subset $B \subseteq A$, the polar

$$B^\perp \equiv \{a : |a| \wedge |b| = 0 \text{ for all } b \in B\}$$

generated by B is a truncation kernel.

Proof. A straightforward induction reveals that $B^\perp = B^{\perp\alpha}$ for all α . \square

2.2.3. Corollary. For any $a \in A^+$, $[a]^* = a^\perp$. Consequently, $[a]^{**} = a^{\perp\perp}$, and it follows that $a \in \overline{a}^{\perp\perp}$, i.e., $a^{\perp\perp} = \overline{a}^{\perp\perp}$.

Proof. Since a^\perp is the pseudocomplement of $\langle a \rangle$ in the lattice $\mathcal{L}A$ of convex ℓ -subgroups of A , the fact that it is a truncation kernel implies that a^\perp must be the pseudocomplement of $[a]$ in the lattice $\mathcal{K}A$ of truncation kernels of A . And we have $a \in [a] = [\bar{a}] \leq [\bar{a}]^{**} = \overline{a}^{\perp\perp}$. \square

2.2.4. Lemma. For convex ℓ -subgroups $A_i \subseteq A$, $[A_1] \cap [A_2] = [A_1 \cap A_2]$.

Proof. Clearly $[A_1] \cap [A_2] \supseteq [A_1 \cap A_2]$, and a simple induction can be used to show that $A_1^\alpha \cap A_2^\alpha \subseteq (A_1 \cap A_2)^\alpha$ for all α . In view of Lemma 2.2.1, this assertion for $\alpha = \omega_1$ establishes that $[A_1] \cap [A_2] \subseteq [A_1 \cap A_2]$. \square

The reader should be warned that Lemma 2.2.4 is false without the hypothesis of convexity.

2.2.5. Lemma. An element $b \in A^+$ lies in a truncation kernel K iff $\overline{nb/n} \in K$ for some $n \in \mathbb{N}$ iff $\overline{nb/n} \in K$ for all $n \in \mathbb{N}$.

Proof. If K is any convex ℓ -subgroup of A then, for any $n \in \mathbb{N}$, $b \in K$ implies $\overline{b/n} \in K$ by convexity since $b \geq b/n \geq \overline{b/n} \geq 0$, with the result that $\overline{nb/n} \in K$ for all n . On the other hand, if K is a truncation kernel and $\overline{nb/n} \in K$ for some $n \in \mathbb{N}$ then $\overline{b/n} \in K$, which implies $b/n \in K$ by Lemma 2.1.2, so that $b \in K$. \square

2.2.6. Lemma. For any $a \in A^+$,

$$[na \ominus 1 : n \in \mathbb{N}] = \left[a \ominus \frac{1}{n} : n \in \mathbb{N} \right] = [a \ominus 0] = [a].$$

Proof. Let $K \equiv [na \ominus 1 : n \in \mathbb{N}]$. We have

$$0 \leq na \ominus 1 = n(a \ominus (1/n)) \leq n(a \ominus 0) = na,$$

so that $K \subseteq [a]$. But since $K = K + na \ominus 1 = K + (na - \overline{na})$, which is to say that $K + na = K + \overline{na}$ for all n , we also have $a \in K$ because A/K must satisfy $(\mathfrak{T}3)$. \square

2.2.7. Corollary. For any $a \in A^+$ and $r \in \mathbb{Q}^+$,

$$\bigvee_{s > r} [a \ominus s] = \left[a \ominus \left(r + \frac{1}{n} \right) : n \in \mathbb{N} \right] = [a \ominus r].$$

Proof. Using Lemma 3.3.6 from [1], express $a \ominus (r + 1/n)$ as $a \ominus r \ominus \frac{1}{n}$, and apply Lemma 2.2.6. \square

2.3. The frame of truncation kernels.

2.3.1. Proposition. \mathcal{KA} is a frame, and the frame operations are

$$K_1 \wedge K_2 = K_1 \cap K_2 \text{ and } \bigvee_I K_i = [K_i : i \in I], \{K_i : i \in I\} \subseteq \mathcal{KA}.$$

The pseudocomplemented elements of \mathcal{KA} are the polars, i.e., $K^* = K^\perp$ for $K \in \mathcal{KA}$. Moreover, for $a_i \in A^+$,

$$[a_1] \cap [a_2] = [a_1 \wedge a_2] \text{ and } [a_1] \cup [a_2] = [a_1 \vee a_2].$$

Proof. Binary meets distribute across arbitrary joins in \mathcal{KA} because the same is true in the frame \mathcal{LA} of convex ℓ -subgroups of A , and the linkage between the two is provided by Lemma 2.2.4. In detail, for truncation kernels $K, K_i, i \in I$, we have

$$K \wedge \bigvee_I K_i = K \cap \left[\bigcup_I K_i \right] = \left[K \cap \bigcup_I K_i \right] = \left[\bigcup_I (K \cap K_i) \right] = \bigvee_I (K \wedge K_i),$$

where \bigcup signifies the join in \mathcal{LA} . Similarly, the displayed equations governing principle truncation kernels hold just because their analogs hold in \mathcal{LA} , i.e., $\langle a_1 \rangle \wedge \langle a_2 \rangle = \langle a_1 \wedge a_2 \rangle$ and $\langle a_1 \rangle \vee \langle a_2 \rangle = \langle a_1 \vee a_2 \rangle$. Finally, from Corollary 2.2.3 we learn that the pseudocomplemented elements of \mathcal{LA} , namely the polars, are present in \mathcal{KA} and therefore constitute the pseudocomplemented elements of \mathcal{KA} . \square

2.3.2. Lemma. *For any $a \in A^+$ we have $[a \ominus r] \prec [a \ominus s]$ for $s < r$ in \mathbb{Q}^+ .*

Proof. It is sufficient to establish this for $s = 0$, for the general case follows by expressing $a \ominus r$ as $(a \ominus s) \ominus (r - s)$ and applying the special case. We want to show that $a \ominus r^\perp \vee [a] = A = \top$. But according to [1, 3.3.7], for each $b \in A^+$ there is some $x \in a \ominus r^\perp$ for which $2(a \vee x) \geq r\bar{b}/r$. It follows that \bar{b}/r lies in $\langle a \ominus r^\perp, a \rangle$, the convex ℓ -subgroup generated by $a \ominus r^\perp \cup \{a\}$. And since the truncation kernel $a \ominus r^\perp \vee [a]$ generated by $a \ominus r^\perp \cup \{a\}$ is absorbing, it must contain b . All this is to say that $a \ominus r^\perp \vee [a] = A = \top$. \square

2.3.3. Lemma. *\mathcal{KA} is regular.*

Proof. By Lemmas 2.2.6 and 2.3.2, each principal truncation kernel $[a]$ can be expressed as a countable join of elements rather below it, viz. $[a] = \bigvee_{\mathbb{N}} [a \ominus (1/n)]$. Since the principal truncation ideals generate \mathcal{KA} , we can be sure it is regular. \square

We write $S_0 \subseteq_{\omega_1} S$ to mean that S_0 is a subset of S which is at most countable.

2.3.4. Lemma. *For any subset $S \subseteq \mathcal{KA}$,*

$$\bigvee S = \bigcup \left\{ \bigvee S_0 : S_0 \subseteq_{\omega_1} S \right\}.$$

Proof. By Lemma 2.1.2 it is sufficient to show that the set displayed on the right, call it K , is an absorbing archimedean kernel. That K is absorbing is clear, so consider elements $a, c \in A^+$ such that $(na - c)^+ \in K$ for all n , say $(na - c)^+ \in \bigvee S_n$ for some $S_n \subseteq_{\omega_1} S$. Put $S_0 \equiv \bigcup_n S_n \subseteq_{\omega_1} S$ and $L \equiv \bigvee S_0 \subseteq K$. Then L is a truncation kernel containing $(na - c)^+$ for all n , hence $a \in L$ by Lemma 2.1.1, hence K is an archimedean kernel by a second application of the same lemma. \square

We summarize the results of our investigation to this point..

2.3.5. Theorem. *\mathcal{KA} is a regular Lindelöf frame.*

Proof. We showed \mathcal{KA} to be regular in Lemma 2.3.3; the fact that it is Lindelöf is an immediate consequence of Lemma 2.3.4. \square

2.4. More about truncation kernels. The functorial representation we seek rests on the properties of certain truncation kernels of A . We develop these properties in this section. This material, however, will not be relevant until the main Section 5. The reader may therefore skip this section without loss of continuity, and refer to these results only as they are put to use.

Two truncation kernels associated with a given $a \in A^+$ play prominent roles in what follows. We define

$$a \blacktriangleright r \equiv [a \ominus r] \text{ and } a \blacktriangleleft r \equiv \bigvee_{0 < s < r} a \ominus s^\perp, \quad r \in \mathbb{Q}^+.$$

The join in the definition of $a \blacktriangleleft r$ is computed in \mathcal{KA} , and can be handily expressed as

$$\bigvee_{s < r} a \blacktriangleright s^* \text{ or } \bigvee_{\mathbb{N}} a \ominus \left(r - \frac{1}{n}\right)^\perp \text{ or } \left[\bigcup_{\mathbb{N}} a \ominus \left(r - \frac{1}{n}\right)^\perp\right].$$

We record the basic properties of these kernels in Lemma 2.4.2, whose proof requires a preliminary observation.

2.4.1. Lemma. *Suppose that $b_1 \succ b_2$ and $c_1 \succ c_2$ in some frame. Then*

$$b_1^* \vee c_1^* \leq (b_1 \wedge c_1)^* \leq b_2^* \vee c_2^*.$$

Proof. The fact that $(b_2^* \vee c_2^*) \vee (b_1 \wedge c_1) = \top$ implies the right inequality. \square

2.4.2. Lemma. *Suppose $a_i \in \overline{A}$ and $0 \leq s \leq r$ in \mathbb{Q} .*

- (1) $a_1 \blacktriangleright r \vee a_2 \blacktriangleright r = (a_1 \vee a_2) \blacktriangleright r$, and $a_1 \blacktriangleright r \wedge a_2 \blacktriangleright r = (a_1 \wedge a_2) \blacktriangleright r$
- (2) $a_1 \blacktriangleleft r \vee a_2 \blacktriangleleft r = (a_1 \wedge a_2) \blacktriangleleft r$, and $a_1 \blacktriangleleft r \wedge a_2 \blacktriangleleft r = (a_1 \vee a_2) \blacktriangleleft r$.
- (3) If $s < r$ then $a \blacktriangleright r \prec a \blacktriangleright s$, hence $a \blacktriangleright s^* \prec a \blacktriangleright r^*$.
- (4) $a \blacktriangleleft s \wedge a \blacktriangleright r = \perp$, and if $s < r$ then $a \blacktriangleleft r \vee a \blacktriangleright s = \top$.
- (5) If $s < r$ then $a \blacktriangleleft s \prec a \blacktriangleleft r$, hence $a \blacktriangleleft r^* \prec a \blacktriangleleft s^*$.

Proof. (1) By Proposition 2.3.1 we can express $a_1 \blacktriangleright r \vee a_2 \blacktriangleright r$ as

$$[a_1 \ominus r] \vee [a_2 \ominus r] = [a_1 \ominus r \vee a_2 \ominus r] = [(a_1 \vee a_2) \ominus r] = (a_1 \vee a_2) \blacktriangleright r.$$

The penultimate equality is provided by [1, 3.3.1(8)]. The argument for the second clause of part (1) is similar.

(2) We can express $a_1 \blacktriangleleft r \wedge a_2 \blacktriangleleft r$ as

$$\begin{aligned} \bigvee_{0 < s_1 < r} [a_1 \ominus s_1]^* \wedge \bigvee_{0 < s_2 < r} [a_2 \ominus s_2]^* &= \bigvee_{0 < s_i < r} ([a_1 \ominus s_1]^* \wedge [a_2 \ominus s_2]^*) = \\ \bigvee_{0 < s_i < r} ([a_1 \ominus s_1] \vee [a_2 \ominus s_2])^* &= \bigvee_{0 < s < r} ([a_1 \ominus s] \vee [a_2 \ominus s])^* = \\ \bigvee_{0 < s < r} [a_1 \ominus s \vee a_2 \ominus s]^* &= \bigvee_{0 < s < r} [(a_1 \vee a_2) \ominus s]^* = (a_1 \vee a_2) \blacktriangleleft r. \end{aligned}$$

And we can express $a_1 \triangleleft r \vee a_2 \triangleleft r$ as

$$\begin{aligned} \bigvee_{\mathbb{N}} a_1 \ominus \left(r - \frac{1}{n}\right)^{\perp} \vee \bigvee_{\mathbb{N}} a_2 \ominus \left(r - \frac{1}{n}\right)^{\perp} &= \\ \bigvee_{\mathbb{N}} \left(a_1 \ominus \left(r - \frac{1}{n}\right)^{\perp} \vee a_2 \ominus \left(r - \frac{1}{n}\right)^{\perp} \right) &= \\ \bigvee_{\mathbb{N}} \left(a_1 \ominus \left(r - \frac{1}{n}\right) \wedge a_2 \ominus \left(r - \frac{1}{n}\right) \right)^{\perp} &= \\ \bigvee_{\mathbb{N}} \left((a_1 \wedge a_2) \ominus \left(r - \frac{1}{n}\right) \right)^{\perp} &= (a_1 \wedge a_2) \triangleleft r \end{aligned}$$

The second equality is justified by Lemma 2.4.2, the third by [1, 3.3.1(7)].

(3) is Lemma 2.3.2. To check (4), compute

$$a \triangleright r \wedge a \triangleleft s = a \triangleright r \wedge \bigvee_{t < s} a \triangleright t^* = \bigvee_{t < s} (a \triangleright r \wedge a \triangleright t^*) = \perp.$$

Then observe that, for $s < r$,

$$a \triangleright s \vee a \triangleleft r = a \triangleright s \vee \bigvee_{t < r} a \triangleright t^* = \bigvee_{s < t < r} (a \triangleright s \vee a \triangleright t^*) = \top.$$

The final equality is implied by (3) above. And finally, (5) holds because $a \triangleright s$ serves as a witness to $a \triangleleft s \prec a \triangleleft r$, a fact we have established in (3) and (4). \square

2.4.3. Lemma. For $a, b \in \overline{A}$, $a + b \in \overline{A}$ implies $b \in a \triangleleft 1$.

Proof. For any n ,

$$a \ominus \left(1 - \frac{1}{n}\right) \wedge b \ominus \frac{1}{n} \leq (a + b) \ominus 1 = 0,$$

by [1, 3.3.7], and this implies that

$$b \ominus \frac{1}{n} \in a \ominus \left(1 - \frac{1}{n}\right)^{\perp} \subseteq \left[\bigcup_{s < 1} a \ominus s^{\perp} \right] = a \triangleleft 1.$$

We conclude that $b \in a \triangleleft 1$ by Lemma 2.2.6. \square

2.4.4. Lemma. For $a, b \in \overline{A}$, $b \triangleright 0 \vee a \triangleleft 1 = (a - b)^+ \triangleleft 1$.

Proof. By replacing b by $a \wedge b$, we may assume without loss of generality that $a \geq b$. The kernel $(a - b)^+ \triangleleft 1$ contains $a \triangleleft 1$ since the map $a \mapsto a \triangleleft 1$ is order reversing by Lemma 2.4.2, and $(a - b)^+ \triangleleft 1$ contains b by Lemma 2.4.3. Consider now a truncation kernel K which contains both $a \triangleleft 1$ and b ; we wish to show that

$$K \supseteq (a - b)^+ \triangleleft 1 = \bigvee_{r < 1} (a - b)^+ \ominus r^{\perp},$$

i.e., $K \supseteq (a - b)^+ \ominus r^\perp = (a - a \wedge b) \ominus r^\perp$ for all $r < 1$. But from [1, 3.3.1(12)] we know that

$$(a - a \wedge b) \ominus r + a \wedge b = a \ominus r \vee (a \wedge b),$$

from which follows $(a - a \wedge b) \ominus r = (a \ominus r - a \wedge b)^+$, so that, in the end, what we need to show is that $(a \ominus r - a \wedge b)^{+\perp} \subseteq K$ for all $r < 1$. For that purpose fix r and consider an element $x \geq 0$ such that $x \wedge (a \ominus r - a \wedge b)^+ = 0$. Note that, since $(a \ominus r - a \wedge b)^+$ may be considered to be the result of disjointifying $a \ominus r$ and $a \wedge b$, it follows that

$$2((a \ominus r - a \wedge b)^+ \vee (a \wedge b)) \geq a \ominus r \vee (a \wedge b).$$

(See the discussion of disjointification preceding [1, 3.3.8].)

We claim that $x \wedge a \ominus r \leq 2(a \wedge b)$, for

$$\begin{aligned} x \wedge a \ominus r &\leq x \wedge 2((a \ominus r - a \wedge b)^+ \vee (a \wedge b)) = \\ &(x \wedge 2(a \ominus r - a \wedge b)^+) \vee (x \wedge 2(a \wedge b)) = x \wedge 2(a \wedge b) \leq 2(a \wedge b). \end{aligned}$$

But then we have

$$\begin{aligned} [x] &= [x] \wedge \top = [x] \wedge (a \blacktriangleright r \vee a \blacktriangleleft 1) = \\ &[x \wedge a \ominus r] \vee ([x] \wedge a \blacktriangleleft 1) \leq [2(a \wedge b)] \vee a \blacktriangleleft 1 \leq [b] \vee a \blacktriangleleft 1 \leq K. \end{aligned}$$

The conclusion is that $x \in K$, and this finishes the proof. \square

For a convex ℓ -subgroup K of A , we let

$${}^0K \equiv \{a \in \overline{A} : a \blacktriangleright 0 \subseteq K\} \text{ and } {}^1K \equiv \{a \in \overline{A} : a \blacktriangleleft 1 \subseteq K\}.$$

0K is a convex monoid with respect to bounded addition, in the sense that $\overline{a + b} \in {}^0K$ whenever $a, b \in {}^0K$. And 1K is a filter on \overline{A} which is disjoint from 0K (if K is proper) by Lemma 2.4.4. In fact, if $K \in \mathcal{KA}$ then 0K is just $\overline{K} = \{\overline{a} : a \in K^+\}$.

2.4.5. Corollary. *Suppose $K \in \mathcal{KA}$. Then $a \in {}^1K$ and $b \in {}^0K$ imply $(a - b)^+ \in {}^1K$*

Proof. This follows directly from Lemma 2.4.4. \square

2.4.6. Corollary. *Suppose $K \in \mathcal{KA}$. Then K contains $a \blacktriangleleft 1$ for some $a \in \overline{A}$ iff $K = \bigcup_{1K} a \blacktriangleleft 1$.*

Proof. Suppose $a \blacktriangleleft 1 \subseteq K$ and $b \in K$ for some $a, b \in \overline{A}$. Then $(a - b)^+ \in {}^1K$ by Corollary 2.4.5, and $b \in (a - b)^+ \blacktriangleleft 1$ by Lemma 2.4.3. \square

3. REPRESENTING A IN \mathcal{RKA}

We exhibit a natural **AT**-injection $A \rightarrow \mathcal{RKA}$. Though intuitive and simple enough, this representation is not functorial. It is, however, the most important step in the development of a fully functorial representation of trunks and their morphisms, culminating in Theorems 5.1.1 and 5.3.1.

3.1. The frame map \underline{a} .

3.1.1. Definition. For $a \in A^+$, define the map $\underline{a} : \{(r, \infty) : r \in \mathbb{Q}\} \rightarrow \mathcal{KA}$ by the rule

$$\underline{a}(r, \infty) \equiv \begin{cases} a \blacktriangleright r, & r \geq 0 \\ \top & r < 0 \end{cases}, r \in \mathbb{Q}.$$

3.1.2. Proposition. Each \underline{a} extends to a unique frame map $\mathcal{O}\mathbb{R} \rightarrow \mathcal{KA}$, which we also denote \underline{a} .

Proof. According to [2, 3.1.2], this amounts to establishing three things.

- $\underline{a}(s, \infty) \prec \underline{a}(r, \infty)$ for $r < s$ in \mathbb{Q} .
- $\underline{a}(r, \infty) = \bigvee_{r < s} \underline{a}(s, \infty)$ for all $r \in \mathbb{Q}$.
- and $\bigvee_{\mathbb{Q}} \underline{a}(r, \infty) = \bigvee_{\mathbb{Q}} \underline{a}(r, \infty)^* = \top$.

The first point is the content of Lemma 2.4.2(3), the second is Corollary 4.5.3, and the third point follows from the claim that $\bigvee_{\mathbb{N}} [a \ominus n]^* = \top$ in \mathcal{KA} . To establish this claim, in turn, it suffices to show that

$$\forall b \in A^+ \exists c \in A^+ \forall n \in \mathbb{N} \exists i \in \mathbb{N} \left((n\bar{b} - c)^+ \wedge a \ominus i = 0 \right).$$

For, when we denote the convex ℓ -subgroup $\bigcup_{\mathbb{N}} [a \ominus n]^*$ by B , the condition displayed above simply asserts that \bar{b} lies in the archimedean kernel generated by B by virtue of satisfying Lemma 2.1.1. And if so, of course, it follows that $b \in [B]$ by Lemma 2.1.2. But satisfying the displayed condition is easy. Given $b \in A^+$, take c to be a , and, upon being presented with n , take i to be n . The condition becomes

$$(n\bar{b} - a)^+ \wedge a \ominus n = n \left((\bar{b} - a/n)^+ \wedge (a/n - \overline{a/n}) \right) = 0.$$

This follows from [1, 3.3.1(1)], with a and b there taken to be a/n and b here. \square

3.1.3. Lemma. For $a \in A^+$ and $0 \leq r < 1$ in \mathbb{Q} , $[\bar{a} \ominus r] = [a \ominus r] = [\overline{a \ominus r}]$.

Proof. In [1, 3.3.5], take p , q , and a there to be r , $1 - r$, and a/r here, to get

$$\begin{aligned} \bar{a} &= r\overline{a/r} + (1 - r) \overline{r \left(a/r - \overline{a/r} \right) / (1 - r)} = \\ r\overline{\bar{a}/r} + (1 - r) \overline{r(a/r \ominus 1) / (1 - r)} &= r\overline{\bar{a}/r} + (1 - r) \overline{(a \ominus r) / (1 - r)}. \end{aligned}$$

This rearranges to $\bar{a} \ominus r = \bar{a} - r\overline{\bar{a}/r} = (1 - r) \overline{(a \ominus r) / (1 - r)}$, hence

$$[\bar{a} \ominus r] = \left[(1 - r) \overline{(a \ominus r) / (1 - r)} \right] = \left[\overline{(a \ominus r) / (1 - r)} \right] = [(a \ominus r) / (1 - r)].$$

Since any nonzero multiple of a generator of a truncation ideal is itself a generator, this works out to $[a \ominus r] = [\bar{a} \ominus r]$. \square

3.1.4. Lemma. For $a \in A^+$ and $r \in \mathbb{F}$,

$$\underline{a}(-\infty, r) = \underline{\underline{a}}(-r, \infty) = \begin{cases} a \blacktriangleleft r, & r > 0 \\ \perp & r \leq 0 \end{cases}$$

Proof. In light of the fact that $a \triangleleft r = \bigvee_{0 < s < r} [a \ominus s]^*$ for $r \geq 0$, this is an application of a general principle ([2, 3.1.1, 3.1.3]): for a frame L , a frame map $f \in \mathcal{RL}$, and for $r \in \mathbb{Q}$,

$$f(-\infty, r) = (-f)(-r, \infty) = \bigvee_{s < r} f(s, \infty)^*.$$

In the present situation, take $f \equiv \underline{a}$. □

We denote the underscore map $a \mapsto \underline{a}$ by $\kappa_A : A \rightarrow \mathcal{RKA}$, and we denote its range by

$$\underline{A} \equiv \{\underline{a} : a \in A\} \subseteq \mathcal{RKA}.$$

We show in Theorem 3.2.2 that κ_A is an isomorphism $A \rightarrow \underline{A}$. A little ground clearing is necessary first.

3.2. κ_A is an isomorphism. We begin by showing that κ preserves truncation and diminution.

3.2.1. Lemma. *For any $a \in A^+$,*

$$\underline{a \ominus 1} = (\underline{a} - 1)^+ \text{ and } \overline{\underline{a}} = \underline{a} \wedge 1.$$

It follows that $\underline{a \ominus r} = (\underline{a} - r)^+$ and $\overline{ra/r} = a \wedge r$ for $r \in \mathbb{Q}$.

Reader beware, for Lemma 3.2.1 can easily be misunderstood. The 1 and the r which appear on the left sides of the first and third equations, respectively, refer to scalars, while the same symbols on the right sides of the first three equations refer to the corresponding constant frame functions $\mathcal{O}\mathbb{R} \rightarrow \mathcal{KA}$. Such a constant function is given by the rule

$$r(s, \infty) = \begin{cases} \perp, & s \geq r \\ \top, & s < r \end{cases}, \text{ or } r(-\infty, s) = \begin{cases} \top, & s > r \\ \perp, & s \leq r \end{cases}.$$

This constant function need not lie in \underline{A} .

Proof. We have, for $r \in \mathbb{Q}$,

$$\underline{a \ominus 1}(r, \infty) = \begin{cases} [(a \ominus 1) \ominus r], & r \geq 0 \\ \top, & r < 0 \end{cases} = \begin{cases} [a \ominus (1 + r)], & r \geq 0 \\ \top, & r < 0 \end{cases}$$

On the other hand we have

$$(\underline{a} - 1)^+(r, \infty) = (\underline{a} - 1)(r, \infty) \vee 0(r, \infty) = \underline{a}(r + 1, \infty) \vee 0(r, \infty),$$

with the equalities justified by 3.1.3 and 3.1.5, respectively, of [2, 3.1.3]. Since

$$\underline{a}(r + 1, \infty) = \begin{cases} [a \ominus (r + 1)], & r + 1 \geq 0 \\ \top, & r + 1 < 0 \end{cases} \text{ and } 0(r, \infty) = \begin{cases} \perp, & r \geq 0 \\ \top, & r < 0 \end{cases},$$

we see by inspection that the $\underline{a \ominus 1}(r, \infty) = (\underline{a} - 1)^+(r, \infty)$ for all $r \in \mathbb{Q}$. From this follows

$$\overline{\underline{a}} = \underline{a} - \underline{a \ominus 1} = \underline{a} - (\underline{a} - 1)^+ = \underline{a} + (1 - \underline{a}) \wedge 0 = \underline{a} \wedge 1. \quad \square$$

3.2.2. Theorem. \underline{A} is a subtrunc of \mathcal{RKA} , and $\kappa_A \equiv (a \mapsto \underline{a})$ is a trunc isomorphism $A \rightarrow \underline{A}$.

Proof. It is folklore that, for ℓ -groups B and C , any map $B^+ \rightarrow C^+$ which preserves meets and sums extends to a unique ℓ -homomorphism $B \rightarrow C$. Therefore we need only check that the restriction of the underscore map to A^+ preserves meets and sums. So consider $a_i \in A^+$ and $r \in \mathbb{F}$.

$$\begin{aligned} (\underline{a_1} \wedge \underline{a_2})(r, \infty) &= \underline{a_1}(r, \infty) \wedge \underline{a_2}(r, \infty) = \begin{cases} [a_1 \ominus r] \wedge [a_2 \ominus r], & r \geq 0 \\ \top, & r < 0 \end{cases} = \\ \begin{cases} [a_1 \ominus r \wedge a_2 \ominus r], & r \geq 0 \\ \top, & r < 0 \end{cases} &= \begin{cases} [(a_1 \wedge a_2) \ominus r], & r \geq 0 \\ \top, & r < 0 \end{cases} = \underline{a_1 \wedge a_2}(r, \infty). \end{aligned}$$

The first equality is justified by [2, 3.1.3], and the fourth by [1, 3.3.1(7)].

Again fix $a_i \in A^+$, and consider

$$(\underline{a_1} + \underline{a_2})(r, \infty) = \bigvee_{U_1 + U_2 \subseteq (r, \infty)} (\underline{a_1}(U_1) \wedge \underline{a_2}(U_2)) = \bigvee_s (\underline{a_1}(s, \infty) \wedge \underline{a_2}(r - s, \infty))$$

The second equality is justified by the observation that if open subsets $U_i \subseteq \mathbb{R}$ satisfy $U_1 + U_2 \subseteq (r, \infty)$ then U_1 must be bounded below, say by s , in which case U_2 must be bounded below by $r - s$. If $s < 0$ then the corresponding term of the join satisfies

$$\underline{a_1}(s, \infty) \wedge \underline{a_2}(r - s, \infty) = \underline{a_2}(r - s, \infty) \leq \underline{a_2}(r, \infty) \leq \underline{a_1 + a_2}(r, \infty),$$

and if $r - s < 0$ then the term satisfies

$$\underline{a_1}(s, \infty) \wedge \underline{a_2}(r - s, \infty) = \underline{a_1}(s, \infty) \leq \underline{a_1}(r, \infty) \leq \underline{a_1 + a_2}(r, \infty).$$

In the only remaining case we have $0 \leq s \leq r$, which gives

$$\begin{aligned} \underline{a_1}(s, \infty) \wedge \underline{a_2}(r - s, \infty) &= [a_1 \ominus s] \wedge [a_2 \ominus (r - s)] = \\ [a_1 \ominus s \wedge a_2 \ominus (r - s)] &\leq [(a_1 + a_2) \ominus r] = \underline{a_1 + a_2}(r, \infty). \end{aligned}$$

The inequality holds by [1, 3.3.7]. Thus have we established that $(\underline{a_1} + \underline{a_2})(r, \infty) \leq \underline{a_1 + a_2}(r, \infty)$.

To establish the opposite inequality it is enough to show that, for any $\varepsilon > 0$ and $r \geq 0$,

$$(\underline{a_1} + \underline{a_2})(r, \infty) \geq \underline{a_1 + a_2}(r + 2\varepsilon, \infty).$$

To that end fix ε and r , and put $x \equiv \underline{a_1 + a_2}(r + 2\varepsilon, \infty)$. Then we have

$$\begin{aligned} x &= x \wedge \top = x \wedge \bigvee_{r_1} \underline{a_1}(r_1 - \varepsilon, r_1 + \varepsilon) = \bigvee_{r_1} (x \wedge \underline{a_1}(r_1 - \varepsilon, r_1 + \varepsilon)) = \\ &\bigvee_{r_1} ((x \wedge \underline{a_1}(-\infty, r_1 + \varepsilon)) \wedge \underline{a_1}(r_1 - \varepsilon, \infty)) \leq \\ &\bigvee_{r_1} (\underline{a_2}(r + \varepsilon - r_1, \infty) \wedge \underline{a_1}(r_1 - \varepsilon, \infty)) = (\underline{a_1} + \underline{a_2})(r, \infty). \end{aligned}$$

The inequality is justified by the observation that

$$\begin{aligned}
 x \wedge \underline{a}_1(-\infty, r_1 + \varepsilon) &= x \wedge \underline{a}_1(-r_1 - \varepsilon, \infty) = \\
 &\quad \underline{a}_1 + \underline{a}_2(r + 2\varepsilon, \infty) \wedge \underline{a}_1(-\infty, r_1 + \varepsilon) = \\
 &\quad \bigvee_{r_2} (\underline{a}_1(r_2 + \varepsilon, \infty) \wedge \underline{a}_2(r + \varepsilon - r_2, \infty)) \wedge \underline{a}_1(-\infty, r_1 + \varepsilon) = \\
 &\quad \bigvee_{r_2} (\underline{a}_1(r_2 + \varepsilon, \infty) \wedge \underline{a}_2(r + \varepsilon - r_2, \infty) \wedge \underline{a}_1(-\infty, r_1 + \varepsilon)) = \\
 &\quad \bigvee_{r_2} (\underline{a}_1(r_2 + \varepsilon, r_1 + \varepsilon) \wedge \underline{a}_2(r + \varepsilon - r_2, \infty)).
 \end{aligned}$$

For if $r_2 \geq r_1$ then the contribution of the corresponding term to the last join is trivial, and if $r_2 < r_1$ then the corresponding term satisfies

$$\underline{a}_1(r_2 + \varepsilon, r_1 + \varepsilon) \wedge \underline{a}_2(r + \varepsilon - r_2, \infty) \leq \underline{a}_2(r + \varepsilon - r_2, \infty) \leq \underline{a}_2(r + \varepsilon - r_1, \infty),$$

with the result that

$$\begin{aligned}
 x \wedge \underline{a}_1(-\infty, r_1 + \varepsilon) &= \bigvee_{r_2} (\underline{a}_1(r_2 + \varepsilon, r_1 + \varepsilon) \wedge \underline{a}_2(r + \varepsilon - r_2, \infty)) \\
 &\leq \underline{a}_2(r + \varepsilon - r_1, \infty).
 \end{aligned}$$

It is obvious that $a \mapsto \underline{a}$ is one-one, since

$$0 < a \in A \implies \underline{a}(0, \infty) \equiv [a] \neq \perp = \underline{0}(0, \infty).$$

Finally, the fact that \rightarrow preserves truncation is the content of Lemma 3.2.1. \square

3.3. The representation $A \rightarrow \mathcal{R}\mathcal{K}A$ is not functorial. To say that the representation κ_A is functorial is to say that $(\kappa_A, \mathcal{K}A)$ constitutes an \mathcal{R} -universal arrow with domain A .

$$\begin{array}{ccc}
 A & \xrightarrow{\kappa_A} & \mathcal{R}\mathcal{K}A \\
 \theta \downarrow & & \downarrow \mathcal{R}g \\
 B & \xrightarrow{\kappa_B} & \mathcal{R}\mathcal{K}B
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathcal{K}A & & \mathcal{O}\mathbb{R} \\
 g \downarrow & \nwarrow \hat{a} & \\
 \mathcal{K}B & \swarrow \theta(a) &
 \end{array}$$

That means that for any other trunc morphism θ , with a codomain of the form $\mathcal{R}L$ for some frame L , there is a frame morphism g such that $\mathcal{R}g \circ \mu = \theta$. But the arrow $(\kappa_A, \mathcal{K}A)$ is no such thing, as we can see from the following simple example

3.3.1. Example. Let $A \equiv \mathbb{R}$, so that $\mathcal{K}A$ is the two-element frame $2 \equiv \{\perp, \top\}$. A little reflection on the definitions leads to the conclusion that, for each $a \in \mathbb{R}$, $\underline{a} = \kappa_A(a)$ is the constant a function, i.e.,

$$\underline{a}(U) = \begin{cases} \top & , \quad a \in U \\ \perp & , \quad a \notin U \end{cases} , \quad U \in \mathcal{O}\mathbb{R}.$$

Now let $B \equiv \mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with cardinal order, so that \mathcal{KB} is the four element frame $4 \equiv \{(\perp, \perp) < (\perp, \top), (\top, \perp) < (\top, \top)\}$. (Here (\perp, \top) represents $0 \times \mathbb{R}$, (\top, \perp) represents $\mathbb{R} \times 0$, etc.) A little more reflection reveals that

$$\underline{(a, b)}(\mathcal{U}) = \begin{cases} (\top, \top) & , \quad a, b \in \mathcal{U} \\ (\perp, \top) & , \quad a \notin \mathcal{U} \ni b \\ (\top, \perp) & , \quad b \notin \mathcal{U} \ni a \\ (\perp, \perp) & , \quad a, b \notin \mathcal{U} \end{cases}, \mathcal{U} \in \mathcal{OR}.$$

Finally, consider the embedding $\theta : A \rightarrow B \equiv (a \mapsto (a, 0))$. Functoriality would require the existence of a frame map $g : \mathcal{KA} \rightarrow \mathcal{KB}$ such that $\mathcal{R}g \circ \kappa_A = \kappa_B \circ \theta$, i.e., such that

$$g \circ \underline{a}(\mathcal{U}) = \underline{\theta(a)}(\mathcal{U}), \mathcal{U} \in \mathcal{OR}.$$

However, there is exactly one frame map $2 \rightarrow 4$, and it is easy to check that it does not have the property displayed.

Note that θ is an example of a trunc morphism between \mathbf{W} -objects which is not a \mathbf{W} -morphism. Thus \mathbf{W} is not a full subcategory of \mathbf{AT} .

It is perhaps surprising that functoriality can be restored to the representation by the simple expedient of adjoining a point to each frame under consideration, and requiring the frame maps to commute with the designated points. Nevertheless this is the case, but to prove it requires a little spadework.

4. POINTED AND FILTERED FRAMES

We digress to fill in the background necessary to systematically adjoin a single point to each frame under consideration. This procedure is not only necessary for our development, but it is also of interest in its own right. We remind the reader of our convention that all frames are completely regular unless explicitly stipulated otherwise. We denote the two-element frame by $2 \equiv \{\perp, \top\}$.

4.1. Pointed frames. A *pointed frame* is a pair $(L, *_L)$, where L is a frame and $* : L \rightarrow 2$ is a point of L . A *pointed frame morphism* $f : (L, *_L) \rightarrow (M, *_M)$ is a frame morphism $f : L \rightarrow M$ which commutes with the points, i.e., such that $*_M \circ f = *_L$. We use \mathbf{pF} to denote the category of pointed frames and their morphisms. Of particular importance is the *frame of the pointed reals* $\mathcal{O}_0\mathbb{R} \equiv (\mathcal{OR}, 0)$, where $0 : \mathcal{OR} \rightarrow 2$ is the constant 0 frame map.

Perhaps the most natural example of a pointed frame is one of the form $2L \equiv (2 \times L, *)$, where L is a frame, $2 \times L$ is the ordinary frame product, and $*$ is the first projection map $(\varepsilon, a) \mapsto \varepsilon$, $\varepsilon = 0, 1$. In fact, we assume the point map to be this projection whenever we deal with a subobject of $2L$.

It is worth mentioning that the second projection

$$\pi : 2L \rightarrow L \equiv ((\varepsilon, a) \mapsto a, a \in L)$$

is the co-free pointed frame over L . More explicitly, let \mathcal{U} be the forgetful functor which assigns to each pointed frame its underlying plain frame. Then, given a pointed frame

$(M, *_M)$ and a frame map $f : \mathcal{U}M \rightarrow L$, there exists a unique pointed frame morphism $k : (M, *_M) \rightarrow 2L$ such that $\pi \circ \mathcal{U}k = f$. In fact, $\mathcal{U}k$ is just the product map $*_M \times f$.

We should also point out that the designated point $*$ of $2L$ is isolated. A point p of a frame M is said to be *isolated* if it is the open quotient of a complemented element. An *isolated point frame* is a pointed frame whose designated point is isolated. Isolated point frames are central to the representation of **W**-objects as trunks; see Section 6. We use **ipF** to denote the full subcategory of **pF** composed of the isolated pointed frames.

4.2. The standard representation of a pointed frame. In any frame M , we define the *kernel* of a point $*$: $M \rightarrow 2$ to be

$$p \equiv \bigvee_{*(a)=\perp} a,$$

the largest element of M sent to \perp by $*$. If M is regular then p is maximal and $*$ is just the closed quotient determined by p . The associated congruence is complemented in the congruence frame of M by the open quotient determined by p , which we denote π . One concrete realization of the two quotients is

$$* : M \rightarrow \uparrow p = (a \mapsto a \vee p) \text{ and } \pi : M \rightarrow \downarrow a = (a \mapsto a \wedge p).$$

If $(M, *_M)$ is a pointed frame then we refer to the kernel of $*_M$ as the *kernel* of M , and write it p_M . We denote the associated closed and open quotient maps by $*_M : M \rightarrow 2$ and $\pi_M : M \rightarrow Q_M$, respectively, and refer to them as canonical. The induced product map $*_M \times \pi_M$, which we denote by $v_M : M \rightarrow 2Q_M$, is a **pF**-morphism by construction, and, since the associated congruences meet to the identity in the congruence lattice, v_M is injective. We drop the subscripts whenever doing so introduces no ambiguity.

$$\begin{array}{ccc} 2Q & \xrightarrow{*} & 2 \\ \pi \downarrow & \swarrow v_M & \uparrow *_M \\ Q & \xleftarrow{\pi_M} & M \end{array}$$

Let L be a frame and F a filter on L . We say that F is *regular* if $\bigvee_F b^* = \top$. Define

$$2_F L \equiv \{(\varepsilon, a) \in 2L : \varepsilon = \top \implies a \in F\}.$$

4.2.1. Lemma. *For any frame L and filter F on L , $2_F L$ is a sub-pointed frame of $2L$ which is regular iff F is regular.*

Proof. Suppose $2_F L$ is regular, and, in order to verify that F is regular, consider $a < \top$ in L . Then $(\perp, a) < p \equiv (\perp, \top)$ in $2_F L$, so that by regularity there must be some $q \prec p$ such that $q \not\leq (\perp, a)$, say $q \wedge r = \perp$ and $r \vee p = \top$ for some $r \in 2_F L$. Now q must be of the form (\perp, b) since $q \leq p$, hence r must be of the form (\top, c) since $q \vee r = \top$. But then the definition of $2_F L$ forces $c \in F$, and $q \wedge r = \perp$ forces $b \wedge c = \perp$, with the result that $c^* \not\leq a$. That is, F is regular.

Now suppose that F is regular. We claim that, for any $c \in L$,

$$(\perp, c) = \bigvee \{(\perp, a \wedge b^*) : b \in F, a \prec c\}.$$

This is true because

$$\begin{aligned} \bigvee_{b \in F, a \prec c} (\perp, a \wedge b^*) &= \left(\perp, \bigvee_{b \in F, a \prec c} (a \wedge b^*) \right) = \left(\perp, \bigvee_{a \prec c} \bigvee_{b \in F} (a \wedge b^*) \right) = \\ &= \left(\perp, \bigvee_{a \prec c} \left(a \wedge \bigvee_{b \in F} b^* \right) \right) = \left(\perp, \bigvee_{a \prec c} a \right) = (\perp, c). \end{aligned}$$

And $(\top, a^* \vee b)$ witnesses $(\perp, a \wedge b^*) \prec (\perp, c)$, for

$$(\top, a^* \vee b) \wedge (\perp, a \wedge b^*) = (\perp, \perp) \text{ and } (\top, a^* \vee b) \vee (\perp, c) = (\top, \top).$$

On the other hand, it is obvious that $(\top, b) = (\top, \bigvee_{a \prec b} a) = \bigvee_{a \prec b} (\top, a)$ for any $b \in F$, and if $a \prec b$ it is just as clear that $(\top, a) \prec (\top, b)$. This shows that $2_F L$ is regular. \square

4.2.2. Proposition. *Let M be a pointed frame with canonical quotient maps $*$ and $\pi : M \rightarrow Q$, and let*

$$F \equiv \{\pi(a) : *(a) = \top\}.$$

Then F is a regular filter on Q , and the range $\nu_M[M]$ of ν_M is $2_F Q$

$$\begin{array}{ccc} 2L & \xrightarrow{*} & 2 \\ \pi \downarrow & \swarrow & \uparrow *_{\mathcal{M}} \\ & 2_F L & \\ \downarrow & \nwarrow & \uparrow \\ L & \xleftarrow{\rho} & M \end{array}$$

We denote the range restriction of ν_M by $\tau_M : M \rightarrow 2_F Q$, and the insertion $2_F Q \rightarrow 2Q$ by σ_M , so that $\nu_M = \sigma_M \tau_M$. We refer to τ_M as the *standard representation* of M .

4.3. The trunc $\mathcal{R}_0 M$. For a pointed frame $(M, *)$, we denote the family of pointed frame morphisms $\mathcal{O}_0 \mathbb{R} \rightarrow M$ by

$$\mathcal{R}_0 M \equiv \text{hom}_{\mathbf{pF}}(\mathcal{O}_0 \mathbb{R}, M) = \{a \in \mathcal{R}M : *_{\mathcal{M}} \circ a = 0\}.$$

4.3.1. Lemma. *$\mathcal{R}_0 M$ is a subtrunc of $\mathcal{R}M$. However, it is not a \mathbf{W} -subobject of $\mathcal{R}M$ if M is nontrivial, i.e., if $\perp < \top$. In fact, the only constant function of $\mathcal{R}M$ present in $\mathcal{R}_0 M$ is the 0 function.*

Proof. For $0 \leq a_i \in \mathcal{R}_0 M$ we have

$$(a_1 + a_2)(r, \infty) = \bigvee_{U_1 + U_2 \subseteq (r, \infty)} (a_1(U_1) \wedge a_2(U_2)) = \bigvee_{r_1 + r_2 = r} (a_1(r_1, \infty) \wedge a_2(r_2, \infty)),$$

for if nonempty open subsets $U_i \subseteq \mathbb{R}$ satisfy $U_1 + U_2 \subseteq (r, \infty)$ then each U_i must be bounded below, and if $r_i = \bigwedge U_i$ then $r_1 + r_2 \geq r$. Therefore

$$*(a_1 + a_2)(r, \infty) = \bigvee_{r_1 + r_2 = r} (*a_1(r_1, \infty) \wedge *a_2(r_2, \infty)) = \begin{cases} \top, & r < 0 \\ \perp, & r \geq 0 \end{cases} = 0.$$

The verifications that $\mathcal{R}_{\mathbf{pFL}}$ is closed under negation, meet, and join all go along similar lines. Finally, for $0 \leq a \in \mathcal{R}_0M$ and $1 \in \mathcal{RM}$ it is clear that $a \wedge 1 \in \mathcal{R}_0M$, since for all $r \in \mathbb{R}$ we have

$$\begin{aligned} * \circ (a \wedge 1)(r, \infty) &= *(a(r, \infty) \wedge 1(r, \infty)) = *a(r, \infty) \wedge *1(r, \infty) \\ &= 0(r, \infty) \wedge 1(r, \infty) = 0(r, \infty) \end{aligned} \quad \square$$

Thanks to Lemma 4.3.1, we can, and do, regard \mathcal{R}_0M as a trunc. In fact, our main result is that objects of this form are universal for **AT**; see Theorems 5.1.1 and 5.3.1. We emphasize that these are virtually never **W**-objects.

On the other hand, the universal **W**-objects are those of the form \mathcal{RL} , L a frame. It is therefore of interest to learn that each such \mathcal{RL} is **AT**-isomorphic to \mathcal{R}_02L .

4.3.2. Proposition. *Let L be a frame, and let $2L \equiv 2 \times L$ be the frame product with projection $\pi : 2L \rightarrow L$. For any $a \in \mathcal{RL}$, let \hat{a} be the induced product map $0 \times a$.*

$$\begin{array}{ccc} 2L & \xrightarrow{*} & 2 \\ \pi \downarrow & \nearrow \hat{a} & \uparrow 0 \\ L & \xleftarrow{a} & 0\mathbb{R} \end{array}$$

*Then $\hat{a} \in \mathcal{R}_02L$, and the hat map $a \mapsto \hat{a}$ effects an **AT**-isomorphism $\mathcal{RL} \rightarrow \mathcal{R}_02L$.*

4.4. Filtered frames. The standard representation $\tau_M : M \rightarrow 2_FQ$ of a pointed frame M suggests that it may be helpful to think of M in terms of Q and F . We are thus led to consider the category of frames with filters.

A *filtered frame* is an object of the form (L, F) , where L is a frame and F is a regular filter on L . A *filtered frame morphism* $f : (L, F) \rightarrow (M, K)$ is a pair (c, f) , where $c \in M$ and $f : L \rightarrow \downarrow c$ is a frame morphism such that

$$a \in F \implies c \rightarrow f(a) \in K, \quad a \in L.$$

Here $c \rightarrow f(a) = \bigvee_{b \wedge f(a) \leq c} b$ is the Heyting arrow operation.

4.4.1. Lemma. *In any frame,*

$$a \leq b \leq c \implies c \rightarrow (b \rightarrow a) \leq b \rightarrow a.$$

Proof. The conclusion follows from the fact that

$$b \wedge (c \rightarrow (b \rightarrow a)) \leq c \wedge (c \rightarrow (b \rightarrow a)) \leq b \rightarrow a,$$

hence

$$b \wedge (c \rightarrow (b \rightarrow a)) \leq b \wedge (b \rightarrow a) \leq a. \quad \square$$

4.4.2. Lemma. *Let $(c, f) : (L, F) \rightarrow (M, K)$ and $(d, g) : (M, K) \rightarrow (N, G)$ be filtered frame morphisms. Then $(g(c), gf) : (L, F) \rightarrow (N, G)$ is a filtered frame morphism.*

Proof. Clearly gf is a frame morphism $L \rightarrow \downarrow g(c)$. Now

$$a \in F \implies c \rightarrow f(a) \in K \implies d \rightarrow g(c \rightarrow f(a)) \in G,$$

and

$$d \rightarrow g(c \rightarrow f(a)) \leq d \rightarrow (g(c) \rightarrow gf(a)) \leq g(c) \rightarrow gf(a).$$

The first inequality is an instance of the rule that $h(u \rightarrow v) \leq h(u) \rightarrow h(v)$ for any frame map h , and the second inequality is an instance of Lemma 4.4.1. \square

We denote the category of filtered frames with their morphisms by **ff**. Of special importance is the *filtered frame of the reals* $(\mathcal{O}(\mathbb{R} \setminus \{0\}), F_0)$, where

$$F_0 \equiv \{U \setminus \{0\} : 0 \in U \in \mathcal{O}\mathbb{R}\},$$

the filter of punctured neighborhoods of 0.

A *fully filtered frame* is a filtered frame (L, F) for which the filter F is improper, i.e., a filtered frame of the form (L, L) . We denote the corresponding full subcategory by **fff**. The point of the next lemma is that

4.4.3. Lemma. *The fully filtered frames comprise a full bireflective subcategory of ff. A reflector for (L, F) is*

$$(L, F) \rightarrow (L, L) = (a \mapsto a).$$

Proof. It is clear that any morphism $(L, F) \rightarrow (K, K)$ factors through this map. \square

4.5. Pointed frames are categorically equivalent to filtered frames.

4.5.1. Lemma. *In any frame,*

$$(b \vee c = \top \text{ and } b \wedge c = d) \implies b = c \rightarrow d.$$

Let $\mathcal{D} : \mathbf{ff} \rightarrow \mathbf{pF}$ be the functor whose action on an object is $(L, F) \mapsto 2_F L$ and whose action on a morphism is

$$(a \mapsto f(a)) \mapsto ((\varepsilon, a) \mapsto (\varepsilon, f(a))),$$

and let $\mathcal{E} : \mathbf{pF} \rightarrow \mathbf{ff}$ be the functor whose action on an object is

$$(M, *_M) \mapsto (\downarrow p_M, p_M \wedge (M \setminus \downarrow p_M))$$

and whose action on a morphism is

$$(g : (M, *_M) \rightarrow (N, *_N)) \mapsto g|_L, \quad L \equiv \downarrow p_M.$$

4.5.2. Proposition. *The functors*

$$\mathbf{ff} \xrightleftharpoons[\mathcal{E}]{\mathcal{D}} \mathbf{pF}$$

constitute a categorical equivalence. In particular, the restrictions of these functors provide a categorical equivalence between fff and ipF. The units of the equivalence are the isomorphisms

$$M \rightarrow 2_F L, \quad L \equiv *_M^{-1}(\perp), \quad p \equiv \bigvee L, \quad F \equiv p \wedge (M \setminus L), \quad (M, *_M) \in \mathbf{pF}, \\ (L, F) \rightarrow (\perp \times L, \perp \times F), \quad (L, F) \in \mathbf{ff},$$

defined by the rules

$$\begin{aligned} a &\longmapsto (*_M(a), p \wedge a), \quad p = \bigvee *_M^{-1}(\perp), \quad a \in M, \\ a &\longmapsto (\perp, a), \quad a \in L. \end{aligned}$$

Proof. This is a straightforward elaboration of Proposition 4.2.2 and its proof. \square

It follows from Lemma 4.4.3 and Proposition 4.5.2 that \mathbf{ipF} is a full bireflective subcategory of \mathbf{pF} . Let us explicitly record the reflector arrow.

4.5.3. Corollary. *The extension*

$$\nu_M : (M, *_M) \rightarrow 2L, \quad L = *_M^{-1}(\perp),$$

*is the free isolated point frame over the pointed frame $(M, *_M)$*

Proof. We pointed out in Proposition 4.2.2 that ν_M factors through $2_F L$, and that the initial factor $M \rightarrow 2_F L$ is a \mathbf{pF} -isomorphism. The final factor $2_F L \rightarrow 2L$ is the extension of Lemma 4.4.3. \square

We offer an example for the reader's edification.

4.5.4. Example. Let \mathfrak{u} be a free ultrafilter on a complete atomless Boolean algebra B , and let M be the pointed frame

$$(2_{\mathfrak{u}}B, *) = \mathcal{D}(B, \mathfrak{u}) = \{(\varepsilon, b) \in 2 \times B : \varepsilon = \top \implies b \in \mathfrak{u}\}.$$

Observe that the only point of M is the designated point $*$, and it is far from isolated. In fact, the free isolated point frame over M is the inclusion $2_{\mathfrak{u}}B \rightarrow 2B = 2 \times B$. The only point of $2B$ is again its designated point, but this time it is isolated. Note that the passage from $2_{\mathfrak{u}}B$ to $2B$ represents a considerable enlargement of the frame.

4.6. The trunc $\mathcal{R}_{\mathbf{ff}}(L, F)$. For a filtered frame (L, F) , we denote the family of filtered frame morphisms $(\mathcal{O}\mathbb{R}, F_0) \rightarrow (L, F)$ by

$$\begin{aligned} \mathcal{R}_{\mathbf{ff}}(L, F) &\equiv \text{hom}_{\mathbf{ff}}((\mathcal{O}\mathbb{R}, F_0), (L, F)) \\ &= \{\alpha \in \mathcal{R}L : \forall \mathfrak{U} \in \mathcal{O}\mathbb{R} \ (0 \in \mathfrak{U} \implies \alpha(\mathfrak{U}) \in F)\}. \end{aligned}$$

It is easy to see that $\mathcal{R}_{\mathbf{ff}}(L, F)$ is isomorphic to a subtrunc of $\mathcal{R}L$.

4.6.1. Corollary. *For a pointed frame M with $\mathcal{E}M \equiv (L, F)$, $\mathcal{R}_{\mathbf{pF}}M$ and $\mathcal{R}_{\mathbf{ff}}(L, F)$ are isomorphic truncs.*

Proof. In the diagram below, we may identify M with $2_F L$ by Proposition 4.2.2. Then the

$$\begin{array}{ccccccc} M & \longrightarrow & 2_F L & \longrightarrow & 2L & \xrightarrow{*} & 2 \\ & & & & \downarrow \pi & \nwarrow \hat{\alpha} & \uparrow 0 \\ & & & & L & \xleftarrow{\alpha} & \mathcal{O}\mathbb{R} \end{array}$$

condition that $\hat{\alpha}$ factors through the insertion $2_F L \rightarrow 2L$ is exactly the condition that α belongs to $\mathcal{R}_{\mathbf{ff}}(L, F)$. \square

We call an element $a \in L$ *cocompact* if, for $S \subseteq L$,

$$a \vee \bigvee S = \top \implies \exists S_0 \subseteq_\omega S \left(a \vee \bigvee S_0 = \top \right).$$

Here the notation $S_0 \subseteq_\omega S$ means that S_0 is a finite subset of S .

Recall that the compactness degree of a frame L is the least regular cardinal κ such that every subset $S \subseteq L$ such that $\bigvee S = \top$ has a subset $S_0 \subseteq_\kappa S$ with $\bigvee S_0 = \top$. We write $\text{comp } L = \kappa$.

4.6.2. Lemma. *Let L be a frame, let F be a filter on L , and let M abbreviate $2_F L$.*

- (1) *When restricted to M , the projection $M \rightarrow L \equiv ((\varepsilon, a) \mapsto a)$ is dense iff F is a proper filter on L .*
- (2) *Suppose $\bigvee_F b^* = \top$. Then M is regular if L is.*
- (3) *$\text{comp } M \leq \text{comp } L$.*
- (4) *If F is contained in the filter of cocompact elements of L then M is compact.*

Proof. (2) We claim that, for any $c \in L$,

$$(\perp, c) = \bigvee \{(\perp, a \wedge b^*) : b \in F, a \prec c\}.$$

This is true because

$$\begin{aligned} \bigvee_{b \in F, a \prec c} (\perp, a \wedge b^*) &= \left(\perp, \bigvee_{b \in F, a \prec c} (a \wedge b^*) \right) = \left(\perp, \bigvee_{a \prec c} \bigvee_{b \in F} (a \wedge b^*) \right) = \\ &= \left(\perp, \bigvee_{a \prec c} \left(a \wedge \bigvee_{b \in F} b^* \right) \right) = \left(\perp, \bigvee_{a \prec c} a \right) = (\perp, c). \end{aligned}$$

And $(\top, a^* \vee b)$ witnesses $(\perp, a \wedge b^*) \prec (\perp, c)$, for

$$(\top, a^* \vee b) \wedge (\perp, a \wedge b^*) = (\perp, \perp) \text{ and } (\top, a^* \vee b) \vee (\perp, c) = (\top, \top).$$

On the other hand, it is obvious that $(\top, b) = (\top, \bigvee_{a \prec b} a) = \bigvee_{a \prec b} (\top, a)$ for any $b \in F$, and if $a \prec b$ it is just as clear that $(\top, a) \prec (\top, b)$. We leave the straightforward proofs of (3) and (4) to the reader. \square

4.7. The spectrum of A . We are finally prepared to introduce the frame canonically associated with A in the functorial representation we seek. The *spectral frame of A* is the frame

$$\mathcal{M}A \equiv 2_F \mathcal{K}A,$$

where F is the filter on $\mathcal{K}A$ generated by the truncation kernels of the form $a \blacktriangleleft 1$, $a \in \overline{A}$. Our use of the letter \mathcal{M} to denote the spectrum is intended to acknowledge the contributions of James Madden, who was responsible in large part for the localic representation in **W** ([6]). We abbreviate $\mathcal{M}A$ to M for the rest of this section.

4.7.1. Theorem. *M is a regular Lindelöf frame.*

Proof. Lemma 4.6.2 is relevant here. By part (3), and in light of Theorem 2.3.5, M is Lindelöf. By part (2), we need only show that

$$\bigvee_{\overline{A}} \left(\bigvee_{0 < r < 1} [(\mathbf{a} \ominus r)]^* \right)^* = \top \text{ in } \mathcal{KA}$$

in order to show that M is regular. Since the identity $(\bigvee \mathbf{a}^*)^* = \bigwedge \mathbf{a}^{**}$ holds in any frame, this amounts to showing that

$$(*) \quad \bigvee_{\overline{A}} \bigwedge_{0 \leq r < 1} [(\mathbf{a} \ominus r)]^{**} = \top.$$

For that purpose fix $\mathbf{a} \in \overline{A}$, $r \in \mathbb{Q}$, and $n \in \mathbb{N}$ such that $r < 1$. From Lemma 2.3.2 we get

$$[n\mathbf{a} \ominus 1] \prec [n\mathbf{a} \ominus r] \implies [n\mathbf{a} \ominus 1] \leq [n\mathbf{a} \ominus r]^{**},$$

and, by letting the r vary, $[n\mathbf{a} \ominus 1] \leq \bigwedge_{0 \leq r < 1} [(\mathbf{a} \ominus r)]^{**}$. Thus, whatever truncation kernel is represented by the left side of the expression in $(*)$, it contains $n\mathbf{a} \ominus 1$ for all n . But then it contains \mathbf{a} by Lemma 2.2.6, and because \mathbf{a} was chosen arbitrarily, it contains all the elements of \overline{A} . Since it must satisfy part (2) of Lemma 2.1.2, it must contain A . We have shown M to be regular. \square

Our plan is to represent A as a subobject of $\mathcal{R}_0\mathcal{MA}$. One important detail remains to be checked.

4.7.2. Lemma. \underline{A} is a subtrunc of $\mathcal{R}_{\text{ff}}\mathcal{KA}$.

Proof. We must show that $\underline{a}(\mathbf{U}) \in F$ for $\mathbf{a} \in A$ and $\mathbf{U} \in \mathcal{O}\mathbb{R}$ such that $0 \in \mathbf{U}$. Without loss of generality we may assume that \mathbf{U} has the form $(-\varepsilon, \varepsilon)$ for some $0 < \varepsilon \in \mathbb{R}$, and, since $\underline{a}(-\varepsilon, \varepsilon) = |\underline{a}|(-\infty, \varepsilon)$, we need only show that $\underline{a}(-\infty, \varepsilon) \in F$ for any $\mathbf{a} \in A^+$. According to Lemma 4.3.1, $\underline{a}(-\infty, \varepsilon) = \bigvee_{0 < s < \varepsilon} [\mathbf{a} \ominus s]^*$. But if we replace s by $r\varepsilon$, we get

$$\underline{a}(-\infty, \varepsilon) = \bigvee_{0 < r < 1} [\mathbf{a} \ominus r\varepsilon]^* = \bigvee_{0 < r < 1} \left[\overline{\mathbf{a}/\varepsilon} \ominus r \right]^* = \overline{\mathbf{a}/\varepsilon} \blacktriangleleft 1 \in F.$$

The second equality is justified by the observation that

$$[\mathbf{a} \ominus r\varepsilon] = [\varepsilon(\mathbf{a}/\varepsilon \ominus r)] = [\mathbf{a}/\varepsilon \ominus r] = \left[\overline{\mathbf{a}/\varepsilon} \ominus r \right]. \quad \square$$

5. THE FUNCTORIAL REPRESENTATION

We have in hand the components of the representation we seek.

$$A \xrightarrow{\mathbf{a} \mapsto \underline{a}} \underline{A} \leq \mathcal{R}_F\mathcal{KA} \xrightarrow{\underline{a} \mapsto \widehat{a}} \mathcal{R}_0\mathcal{MA}$$

Combining these components results in Theorem 5.1.1, which summarizes the development to this point.

5.1. The representation of objects.

5.1.1. Theorem. For $a \in A^+$, define

$$\hat{a}(r, \infty) \equiv \begin{cases} (\perp, a \blacktriangleright r), & r \geq 0 \\ (\top, \top), & r < 0 \end{cases}.$$

Then \hat{a} extends to a unique element $\hat{a} \in \mathcal{R}_p\mathcal{MA}$, and the map $a \mapsto \hat{a}$, $a \in A^+$, extends to a unique truncation isomorphism

$$\mu_A : A \rightarrow \hat{A} \equiv \{\hat{a} : a \in A\} \leq \mathcal{R}_p\mathcal{MA}.$$

We turn now to the issue of functoriality. This requires a few pertinent facts about subtruncs of \mathcal{RL} , for L a frame.

5.2. Cozero facts. Throughout this subsection L designates a frame and A designates a subtrunc of \mathcal{RL} . Let us recall some standard terminology relevant to this situation. The *cozero element* of $a \in A$ is

$$\text{coz } a \equiv a((-\infty, 0) \cup (0, \infty)).$$

In similar spirit we define the *co-one element* of a to be

$$\text{con } a \equiv a((-\infty, 1) \cup (1, \infty)).$$

If $a \in \bar{A}$ then these expressions simplify to $\text{coz } a = a(0, \infty)$ and $\text{con } a = (-\infty, 1)$. Finally, we will frequently and without comment use the fact that $\text{con } a = \bigvee_{s < 1} a(s, \infty)^*$.

5.2.1. Lemma. For any subset $S \subseteq A$, $\bigvee_S \text{coz } a = \bigvee_{[S]} \text{coz } a$.

Proof. By Lemma 2.2.1, we need only show that $\bigvee_S \text{coz } a = \bigvee_{S^\alpha} \text{coz } a$ for all α , and this we do by induction. If $\alpha = 0$ then $S^\alpha = \langle S \rangle$, the convex ℓ -subgroup of A generated by S . Keeping in mind the facts that $\text{coz } a = \text{coz } |a|$, $0 \leq a \leq b$ implies $\text{coz } a \leq \text{coz } b$, and $\text{coz } |a \vee b|, \text{coz } |a + b| \leq \text{coz } |a| \vee \text{coz } |b|$, the truth of the assertion is clear in this case. Assume now that the assertion holds for all $\gamma < \alpha$. If α is a limit ordinal then the assertion clearly also holds at α , so assume α is of the form $\gamma + 1$. If $\gamma \equiv 0 \pmod 3$ and $b \in S^{\alpha+}$ then there is some $c \in S^{\gamma+}$ for which $(nb - c)^+ \in S^\gamma$ for all n . But since $\text{coz } b = \bigvee_{\mathbb{N}} \text{coz } (nb - c)^+$, it follows that $\text{coz } b \leq \bigvee_{S^\gamma} \text{coz } a$, with the result that

$$\bigvee_{S^\alpha} \text{coz } a = \bigvee_{S^\gamma} \text{coz } a = \bigvee_S \text{coz } a.$$

The argument for the case in which $\gamma \equiv 1 \pmod 3$ is almost identical, and, in light of the fact that $\text{coz } \bar{a} = \text{coz } a$ for $a \in A^+$, the argument for the case in which $\gamma \equiv 2 \pmod 3$ is trivial. \square

5.2.2. Lemma. For $a, b \in A^+$ with $a \in \bar{A}$ and $b \in a \blacktriangleleft 1$, $\text{coz } b \leq \text{con } a$. Therefore $\text{con } a \geq \bigvee_{a \blacktriangleleft 1} \text{coz } b$.

Proof. Abbreviate $\bigcup_{\mathbb{N}} a \ominus (1 - 1/n)^\perp$ to K . Since

$$a \blacktriangleleft 1 \equiv \bigvee_{\mathbb{N}} \left[a \ominus \left(1 - \frac{1}{n} \right) \right]^* = \bigvee_{\mathbb{N}} a \ominus \left(1 - \frac{1}{n} \right)^\perp = [K] = K^{\omega_1}$$

by Propositions 4.5.2 and 2.3.1, it is sufficient to demonstrate that, for all α , $\text{coz } b \leq \text{con } a$ whenever $b \in K^{\alpha+}$. This we do by induction on α . If $\alpha = 0$ then we would have $b \wedge a \ominus (1 - 1/n) = 0$ for some n , with the result that

$$\begin{aligned} \perp &= \text{coz } 0 = \text{coz} \left(b \wedge a \ominus \left(1 - \frac{1}{n} \right) \right) = \\ &\text{coz } b \wedge \text{coz} \left(a - \left(1 - \frac{1}{n} \right) \right)^+ = \text{coz } b \wedge a \left(1 - \frac{1}{n}, \infty \right). \end{aligned}$$

Therefore

$$\top = \text{con } a \vee a \left(1 - \frac{1}{n}, \infty \right) \implies \text{coz } b \leq \text{con } a.$$

Now assume the assertion holds for all γ , $\gamma < \alpha < \omega_1$. If α is a limit ordinal then the assertion holds also at α , so assume $\alpha = \gamma + 1$ for some γ . If $\gamma \equiv 0 \pmod 3$ then there is some $c \in K^{\gamma+}$ such that $(nb - c)^+ \in K^\gamma$ for all n . By the inductive hypothesis we have $\text{coz}(nb - c)^+ \leq \text{con } a$ for all n . Now

$$\text{coz}(nb - c)^+ = (nb - c)(0, \infty) = \bigvee_{nU_1 - U_2 \subseteq (0, \infty)} (b(U_1) \wedge c(U_2))$$

But if open subsets $U_i \subseteq \mathbb{R}$ satisfy $nU_1 - U_2 \subseteq (0, \infty)$ then U_1 must be bounded below, say by t_1 , and U_2 must be bounded above, say by t_2 , where $nt_1 - t_2 > 0$. That is to say that, in the last supremum displayed above, U_1 and U_2 may be replaced by (t, ∞) and $(-\infty, nt)$ for some $t \in \mathbb{R}$. This gives

$$\begin{aligned} \text{con } a &\geq \bigvee_n \bigvee_t (b(t, \infty) \wedge c(-\infty, nt)) = \\ &\bigvee_t \bigvee_n (b(t, \infty) \wedge c(-\infty, nt)) = \bigvee_t \left(b(t, \infty) \wedge \bigvee_n c(-\infty, nt) \right). \end{aligned}$$

But $\bigvee_n c(-\infty, nt) = \perp$ for $t \leq 0$ since $c \geq 0$ and $0(-\infty, 0) = \perp$, while $\bigvee_n c(-\infty, nt) = \top$ for $t > 0$. Therefore the last expression displayed above works out to $\bigvee_{t>0} b(t, \infty) = b(0, \infty) = \text{coz } b$, as desired.

Consider next the case in which $\gamma \equiv 1 \pmod 3$. By the inductive hypothesis we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} \text{con } a &\geq \text{coz } nb \ominus 1 = \text{coz}(nb - 1)^+ = (nb - 1)(0, \infty) \\ &= nb(1, \infty) = b\left(\frac{1}{n}, \infty\right), \end{aligned}$$

with the result that $b(0, \infty) = \bigvee_n b(1/n, \infty) \leq \text{con } a$. In the last case $\gamma \equiv 2 \pmod 3$, and $\text{coz } \overline{b} \leq \text{con } a$ by the inductive hypothesis. But

$$\text{coz } \overline{b} = (b \wedge 1)(0, \infty) = b(0, \infty) = \text{coz } b,$$

and so the proof is complete. \square

5.2.3. Corollary. For any $K \in \mathcal{KA}$, $\bigvee_{1_K} \text{con } a \geq \bigvee_{0_K} \text{coz } b = \bigvee_K \text{coz } b$.

Proof. We have

$$\bigvee_{1_K} \text{con } a \geq \bigvee_{a \in 1_K} \bigvee_{b \in a \triangleleft 1} \text{coz } b = \bigvee_K \text{coz } b.$$

The equality holds because $a \in 1_K$ means that $a \triangleleft 1 \subseteq K$, and because $K = \bigcup_{1_K} a \triangleleft 1$. \square

5.2.4. Lemma. For $a, b \in \overline{A}$ with $a \in \overline{A}$, $\text{coz } b \wedge \text{con } a \leq \bigvee_{a \triangleleft 1} \text{coz } c$.

Proof. Because

$$\text{coz } b = \text{coz } b \wedge \top = \text{coz } b \wedge \text{coz } \frac{1}{2} = \text{coz } \left(b \wedge \frac{1}{2} \right) = \text{coz } \left(\frac{1}{2} \overline{2b} \right),$$

we may assume that $b = (1/2) \overline{2b}$. Since $\text{con } a = \bigvee_{\mathbb{N}} a(1 - 1/n, \infty)^*$, it is sufficient to show that for each $n \in \mathbb{N}$ there exists $c \in a \triangleleft 1^+$ such that

$$\text{coz } b \wedge a(1 - 1/n, \infty)^* \leq \text{coz } c.$$

Fix n , and put $a_1 \equiv na \ominus (n - 1)$. Note that $a_1 \in \overline{A}$ because

$$a \ominus 1 = na \ominus (n - 1) \ominus 1 = na \ominus na = n(a \ominus 1) = 0.$$

Let $c \equiv (b - a_1)^+$, and observe that

$$* \quad c \wedge (a_1 - b)^+ = (b - a_1)^+ \wedge (a_1 - b)^+ = 0,$$

and $2(a_1 \vee c) \geq b$. By Corollary 2.2.3(3),

$$\begin{aligned} (a_1 - b)^+ &\geq a_1 \ominus \frac{1}{2} = na \ominus (n - 1) \ominus \frac{1}{2} = \\ &na \ominus \left(na - \frac{1}{2} \right) = n \left(a \ominus \left(1 - \frac{1}{2n} \right) \right). \end{aligned}$$

Combined with (*), this yields $c \wedge n(a \ominus (1 - \frac{1}{2n})) = 0$, hence $c \wedge a \ominus (1 - \frac{1}{2n}) = 0$, i.e., $c \in a \ominus (1 - 1/2)^{\perp} \subseteq a \triangleleft 1$. With the aid of Lemma 3.2.1 we now get

$$\text{coz } a_1 = \text{coz } na \ominus (n - 1) = \text{coz } (na - (n - 1))^+ = (na)(n - 1, \infty) = a \left(1 - \frac{1}{n}, \infty \right)$$

From the inequality in (*) comes the information that

$$\begin{aligned} \text{coz } b &\leq \text{coz } 2(a_1 \vee c) = \text{coz } (a_1 \vee c) \\ &= \text{coz } a_1 \vee \text{coz } c = a \left(1 - \frac{1}{n}, \infty \right) \vee \text{coz } c. \end{aligned}$$

If we now meet both sides with $a(1 - 1/n)^*$ we get

$$\text{coz } b \wedge a \left(1 - \frac{1}{n}\right)^* = \text{coz } c \wedge a(1 - 1/n)^* \leq \text{coz } c. \quad \square$$

5.2.5. Proposition. For $a, b \in A^+$ with $a \in \overline{A}$,

$$\text{coz } b \wedge \text{con } a = \text{coz } b \wedge \bigvee_{a \triangleleft 1} \text{coz } c = \bigvee_{c \in a \triangleleft 1} \text{coz } (b \wedge c).$$

Proof. According to Lemma 5.2.2, $\text{con } a \geq \bigvee_{a \triangleleft 1} \text{coz } c$. And according to Lemma 5.2.4, $\text{coz } b \wedge \text{con } a \leq \bigvee_{a \triangleleft 1} \text{coz } c$. Together, these two facts imply the equality asserted in the proposition. \square

5.2.6. Proposition. For $a, b \in \overline{A}$, $\text{con } a \vee \bigvee_{b \triangleleft 1} \text{coz } c \geq \text{con } b$.

Proof. We begin with the observation that, by Lemma 2.4.2, $b \triangleleft 1 \vee b \triangleright r = \top$ in \mathcal{KA} for $r < 1$. Therefore, if we let $I \equiv b \triangleleft 1 \cup \{b \ominus r\}$, we can say that

$$\text{coz } b \ominus r \vee \bigvee_{b \triangleleft 1} \text{coz } c = \bigvee_C \text{coz } c = \bigvee_{[C]} \text{coz } c = \bigvee_A \text{coz } c.$$

Consequently $\text{coz } b \ominus r \vee \bigvee_{b \triangleleft 1} \text{coz } c \geq \text{coz } a$, with the result that

$$\text{con } a \vee \text{coz } b \ominus r \vee \bigvee_{b \triangleleft 1} \text{coz } c \geq \text{coz } a \vee \text{con } a = a(0, \infty) \vee a(-\infty, 1) = \top.$$

But this implies that

$$\text{con } a \vee \bigvee_{b \triangleleft 1} \text{coz } c \geq (\text{coz } b \ominus r)^* = b(r, \infty)^*$$

for $r < 1$, and, since $\text{con } b = \bigvee_{r < 1} b(r, \infty)^*$, this proves the proposition. \square

5.3. The representation of morphisms. We show that \mathcal{R}_p is adjoint, which is to say that (μ_A, \mathcal{MA}) is an \mathcal{R}_p -universal arrow with domain A .

5.3.1. Theorem. For any trunc morphism $\theta : A \rightarrow \mathcal{R}_p L$ there is a unique pointed frame morphism g such that $\mathcal{R}_p g \circ \mu_A = \theta$.

$$\begin{array}{ccc} A & \xrightarrow{\mu_A} & \mathcal{R}_0 \mathcal{MA} \\ & \searrow \theta & \downarrow \mathcal{R}_0 g \\ & & \mathcal{R}_0 L \end{array} \quad \begin{array}{ccc} \mathcal{MA} & \xleftarrow{\hat{a}} & \mathcal{O}_0 \mathbb{R} \\ & \searrow & \leftarrow 2 \\ g \downarrow & \nearrow & \leftarrow \theta(a) \\ L & & \end{array}$$

Proof. Observe that $g(\perp, a \blacktriangleright 0) = g\hat{a}(0, \infty) = \theta(a)(0, \infty) = \text{coz } \theta(a)$ for any $a \in \overline{A}$, from which it follows that

$$\begin{aligned} g(\perp, K) &= g\left(\perp, \bigvee_{K^0} a \blacktriangleright 0\right) = g\left(\bigvee_{K^0} (\perp, a \blacktriangleright 0)\right) = \\ &= \bigvee_{K^0} g(\perp, a \blacktriangleright 0) = \bigvee_{K^0} \theta(a)(0, \infty) = \bigvee_{K^0} \text{coz } \theta(a). \end{aligned}$$

Likewise $g(\top, a \blacktriangleleft 1) = g\hat{a}(-\infty, 1) = \theta(a)(-\infty, 1) = \text{con } \theta(a)$ for any $a \in \overline{A}$, from which it follows that

$$\begin{aligned} g(\top, K) &= g\left(\top, \bigvee_{K^1} a \blacktriangleleft 1\right) = g\left(\bigvee_{K^1} (\top, a \blacktriangleleft 1)\right) = \\ &= \bigvee_{K^1} g(\top, a \blacktriangleleft 1) = \bigvee_{K^1} \theta(a)(-\infty, 1) = \bigvee_{K^1} \text{con } \theta(a). \end{aligned}$$

Therefore we have no choice but to define

$$g(\varepsilon, K) \equiv \begin{cases} \bigvee_{K^0} \text{coz } \theta(a), & \varepsilon = \perp \\ \bigvee_{K^1} \text{con } \theta(a), & \varepsilon = \top \end{cases}, \quad K \in \mathcal{KA}, \varepsilon \in 2.$$

Clearly $g(\perp, 0) = \perp$, and

$$A = 0 \blacktriangleleft 1 \implies 0 \in A^1 \implies g(\top, A) \geq 0(-\infty, 1) = \top.$$

The proof is completed by showing that g preserves binary meets and arbitrary joins. This we do in a sequence of lemmas, all phrased in the notation above. \square

5.3.2. Lemma. *g preserves binary meets.*

Proof. Consider $(\varepsilon_i, K_i) \in M$. In the first case $\varepsilon_0 = \varepsilon_1 = \perp$, so we have

$$\begin{aligned} g(\perp, K_0) \wedge g(\perp, K_1) &= \bigvee_{K_0^0} \text{coz } \theta(a_0) \wedge \bigvee_{K_1^0} \text{coz } \theta(a_1) = \bigvee_{a_i \in K_i^0} (\text{coz } \theta(a_0) \wedge \text{coz } \theta(a_1)) = \\ &= \bigvee_{K_i^0} \text{coz } \theta(a_0 \wedge a_1) = \bigvee_{(K_0 \wedge K_1)^0} \text{coz } \theta(a) = g(\perp, K_0 \wedge K_1) = g((\perp, K_0) \wedge (\perp, K_1)). \end{aligned}$$

In the second case $\varepsilon_0 = \varepsilon_1 = \top$, and the argument goes along similar lines. In the third and last case $\varepsilon_0 = \perp < \top = \varepsilon_1$, so we have

$$g(\perp, K_0) \wedge g(\top, K_1) = \bigvee_{K_0^0} \text{coz } \theta(a_0) \wedge \bigvee_{K_1^1} \text{con } \theta(a_1) = \bigvee_{a_i \in K_i^1} (\text{coz } \theta(a_0) \wedge \text{con } \theta(a_1))$$

By Proposition 5.2.5,

$$\begin{aligned} \bigvee_{a_i \in K_i^i} (\text{coz } \theta(a_0) \wedge \text{con } \theta(a_1)) &= \bigvee_{a_i \in K_i^i} \left(\text{coz } \theta(a_0) \wedge \bigvee_{c \in a_1 \blacktriangleleft 1} \text{coz } \theta(c) \right) = \\ &= \bigvee_{a_i \in K_i^i} \bigvee_{c \in a_1 \blacktriangleleft 1} (\text{coz } \theta(a_0) \wedge \text{coz } \theta(c)). \end{aligned}$$

At this point it is useful to remind the reader that $K_1^1 \equiv \{a \in \overline{A} : a \blacktriangleleft 1 \subseteq K_1\}$, so that

$$\begin{aligned} \bigvee_{a_i \in K_i^i} \bigvee_{c \in a_1 \blacktriangleleft 1} (\text{coz } \theta(a_0) \wedge \text{coz } \theta(c)) &= \bigvee_{a_i \in K_i^i} (\text{coz } \theta(a_0) \wedge \text{coz } \theta(a_1)) = \\ &= \bigvee_{a \in (K_1 \wedge K_2)^0} \text{coz } \theta(a) = g(\perp, K_1 \wedge K_2) = g((\perp, K_1) \wedge (\perp, K_2)) \end{aligned} \quad \square$$

5.3.3. Lemma. *g preserves all joins of the form*

$$\begin{aligned} \bigvee_I (\perp, K_i) &= \left(\perp, \bigvee_I K_i \right), \quad (\perp, K_i) \in \mathcal{MA}, \text{ or} \\ \bigvee_I (\top, K_i) &= \left(\top, \bigvee_I K_i \right), \quad (\top, K_i) \in \mathcal{MA}. \end{aligned}$$

Proof. Let $K \equiv \bigvee_I K_i$. We have

$$\begin{aligned} \bigvee_I g(\perp, K_i) &= \bigvee_I \bigvee_{a_i \in {}^0 K_i} \text{coz } \theta(a_i) = \bigvee_I \bigvee_{a_i \in K_i} \text{coz } \theta(a_i) = \bigvee_{a \in \bigcup K_i} \text{coz } \theta(a) = \\ &= \bigvee_{a \in [K_i]} \text{coz } \theta(a) = \bigvee_K \text{coz } \theta(a) = g(\perp, K) = g\left(\bigvee_I (\perp, K_i)\right) \end{aligned}$$

On the other hand,

$$\begin{aligned} \bigvee_I g(\top, K_i) &= \bigvee_I \bigvee_{a_i \in {}^1 K_i} \text{con } \theta(a_i) = \bigvee_{a \in \bigcup {}^1 K_i} \text{con } \theta(a) \leq \\ &= \bigvee_{{}^1 K} \text{con } \theta(a) = g(\top, K) = g\left(\bigvee_I (\top, K_i)\right). \end{aligned}$$

To prove the opposite inequality, observe first that, by Lemma 5.2.2,

$$\begin{aligned} \bigvee_I g(\top, K_i) &= \bigvee_I \bigvee_{a_i \in {}^1 K_i} \text{con } \theta(a_i) \geq \bigvee_I \bigvee_{a_i \in K_i} \text{coz } \theta(a_i) = \\ &= \bigvee_{a \in \bigcup K_i} \text{coz } \theta(a) = \bigvee_{a \in [K_i]} \text{coz } \theta(a) = \bigvee_K \text{coz } \theta(a). \end{aligned}$$

Fix $i_0 \in I$ and $a_0 \in {}^1K_{i_0}$, so that

$$\bigvee_I g(\top, K_i) \geq g(\top, K_{i_0}) = \bigvee_{{}^1K_{i_0}} \text{con } \theta(a) \geq \text{con } \theta(a_0).$$

Now consider an arbitrary $c \in {}^1K$, for which we would have $\bigvee_K \text{coz } \theta(a) \geq \bigvee_{c \triangleleft 1} \text{coz } \theta(a)$ since $c \triangleleft 1 \subseteq K$. We then get from Proposition 5.2.6 that

$$\bigvee_I g(\top, K_i) \geq \text{con } \theta(a_0) \vee \bigvee_K \text{coz } \theta(a) \geq \text{con } \theta(a_0) \vee \bigvee_{c \triangleleft 1} \text{coz } \theta(a) \geq \text{con } c.$$

Since c was chosen arbitrarily, we have proven that $\bigvee_I g(\top, K_i) \geq \bigvee_{{}^1K} \text{con}$, which is to say that we have proven the lemma. \square

5.3.4. Lemma. *g preserves binary joins of the form*

$$(\perp, K_0) \vee (\top, K_1) = (\top, K_1 \vee K_2), \quad (\perp, K_0), (\top, K_1) \in \mathcal{MA}.$$

Proof. To this join g assigns the frame element $g(\top, K) = \bigvee_{{}^1K} \text{con } \theta(a)$, where $K \equiv K_1 \vee K_2$. We should compare this to

$$\begin{aligned} g(\perp, K_0) \vee g(\top, K_1) &= \left(\bigvee_{{}^0K_0} \text{coz } \theta(b) \right) \vee \left(\bigvee_{{}^1K_1} \text{con } \theta(a) \right) = \\ &= \bigvee_{b \in {}^0K_0} \bigvee_{a \in {}^1K_1} (\text{coz } \theta(b) \vee \text{con } \theta(a)) = \bigvee_{b \in {}^0K_0} \bigvee_{a \in {}^1K_1} \text{con } \theta(a - b)^+. \end{aligned}$$

Now $b \in {}^0K_0$ means that $b \triangleright 0 \subseteq K_0$ and $a \in {}^1K_1$ means that $a \triangleleft 1 \subseteq K_1$, so by Lemma 2.4.4 we get

$$(a - b)^+ \triangleleft 1 = b \triangleright 0 \vee a \triangleleft 1 \subseteq K_0 \vee K_1.$$

In other words $(a - b)^+ \in {}^1K$, with the consequence that

$$\bigvee_{b \in {}^0K_0} \bigvee_{a \in {}^1K_1} \text{con } \theta(a - b)^+ \leq \bigvee_{{}^1K} \text{con } \theta(c).$$

Proposition 5.2.6 provides the key step in the proof of the opposite inequality. Fix some $a \in {}^1K_1$, and remember that $g(\top, K_1) \geq \text{con } \theta(a)$. Consider any $b \in {}^1K$, and remember that $g(\perp, K_0) \geq \bigvee_{b \triangleleft 1} \text{coz } \theta(c)$. Then see that

$$\begin{aligned} g(\perp, K_0) \vee g(\top, K_1) &\geq \text{con } \theta(a) \vee g(\perp, K_0) \vee g(\perp, K_1) = \\ &= \text{con } \theta(a) \vee \bigvee_{{}^0K} \text{coz } \theta(c) \geq \text{con } \theta(a) \vee \bigvee_{b \triangleleft 1} \text{coz } \theta(c) \geq \text{con } \theta(b). \end{aligned}$$

Since b was chosen arbitrarily, we get $g(\perp, K_0) \vee g(\top, K_1) \geq \bigvee_{{}^1K} \text{con } \theta(c)$, as desired. \square

5.3.5. Lemma. *g preserves all joins.*

Proof. Consider the join $\bigvee_I (\varepsilon_i, K_i) = (\varepsilon, \bigvee_I K_i)$, and let $I_0 \equiv \{i \in I : \varepsilon_i = \perp\}$ ($I_1 \equiv \{i \in I : \varepsilon_i = \top\}$). Then this join can be parsed as

$$\bigvee_I (\varepsilon_i, K_i) = \bigvee_{I_0} (\varepsilon_i, K_i) \vee \bigvee_{I_1} (\varepsilon_i, K_i),$$

and, by Lemmas 5.3.3 and 5.3.4, g preserves the joins on the right. \square

The proof of Theorem 5.3.1 is complete.

6. \mathbf{W} IS MONOREFLECTIVE IN \mathbf{AT}

In this section we establish that \mathbf{W} is monoreflective in \mathbf{AT} , i.e., that every \mathbf{AT} -object is the domain of a \mathbf{W} -universal arrow.

6.1. Characterizing \mathbf{W} -objects.

6.1.1. Proposition. *The following are equivalent for a trunc A .*

- (1) *A lies in \mathbf{W} , i.e., A^+ contains an element a_0 such that $\bar{a} = a \wedge a_0$ for all $a \in A^+$.*
- (2) *There is some element $a_0 \in \bar{A}$ for which $a_0 \blacktriangleleft 1 = 0$.*
- (3) *$\mathcal{M}A \in \mathbf{ipF}$, i.e., the designated point of $\mathcal{M}A$ is isolated.*
- (4) *\bar{A} contains a greatest element.*

Proof. (1) implies (2). Suppose A^+ contains an element a_0 such that $\bar{a} = a \wedge a_0$ for all $a \in A^+$. Observe that $[a_0] = A$, for $[a_0]$ contains $\bar{a} = a \wedge a_0$ for all $a \in A^+$, and $[a_0]$ has the closure property of Lemma 2.1.2(2). Therefore, for $s < 1$ in \mathbb{F} we have

$$[a_0 \ominus s] = \left[s \left(\frac{a_0}{s} \ominus 1 \right) \right] = \left[s \left(\frac{a_0}{s} - a_0 \right)^+ \right] = [(1-s) a_0] = [a_0] = A.$$

The second equality is a consequence of the fact that, since truncation in A is given by meet with a_0 , diminution is given by the rule $a \ominus 1 = (a - a_0)^+$, $a \in A^+$. The point is that

$$a_0 \blacktriangleleft 1 \equiv \bigvee_{0 < s < 1} [a_0 \ominus s]^* = 0,$$

the bottom element of $\mathcal{K}A$.

(2) is equivalent to (3). According to the categorical equivalence between \mathbf{pF} and \mathbf{fF} outlined in Proposition 4.5.2, $\mathcal{M}A$ is isolated iff $\mathcal{D}\mathcal{M}A = (\mathcal{K}A, F)$ is fully filtered, i.e. iff the filter F used to define $\mathcal{M}A$ from $\mathcal{K}A$ is improper, meaning $0 \in F$. But F is generated by truncation kernels of the form $a \blacktriangleleft 1$, $a \in \bar{A}$.

(2) implies (4). Suppose that, for some $a_0 \in \bar{A}$, we have

$$0 = a_0 \blacktriangleleft 1 = \bigvee_{s < 1} [a_0 \ominus s]^*.$$

It follows that $[a_0 \ominus s] = A$ for all $s < 1$. According to Lemma 3.37 of [1],

$$a_0 \ominus s \wedge (a - a_0)^+ \ominus (1-s) \leq (a \vee a_0) \ominus 1 = 0$$

for all $a \in \overline{A}$. Therefore

$$[a_0 \ominus s] \wedge [(a - a_0)^+ \ominus (1 - s)] = 0.$$

Since $[a_0 \ominus s] = A$, it follows that $[(a - a_0)^+ \ominus (1 - s)] = 0$. From Lemma 2.2.6 we then get

$$[(a - a_0)^+] = \bigvee_{n \in \mathbb{N}} \left[(a - a_0)^+ \ominus \frac{1}{n} \right] = 0,$$

with the result that $(a - a_0)^+ = 0$, i.e., $a_0 \geq a$.

(4) implies (1). Suppose that \overline{A} contains greatest element \overline{b} . Then for any $a \in A^+$ we have $a \wedge \overline{b} \leq \overline{a}$ by axiom (T1). But $\overline{a} \leq a$ by the same axiom, and $\overline{a} \leq \overline{b}$ by hypothesis, with the result that $\overline{a} \leq a \wedge \overline{b}$. In short, (1) holds. \square

6.1.2. Proposition. *Let A be a trunc in \mathbf{W} , and let a_0 be the largest element of \overline{A} . Then, in $\widehat{A} = \mu_A[A]$, we have*

$$\widehat{a_0}(u) = \begin{cases} (\top, \top) & , \quad 0, 1 \in u \\ (\top, \perp) & , \quad 1 \notin u \ni 0 \\ (\perp, \top) & , \quad 0 \notin u \ni 1 \\ (\perp, \perp) & , \quad 0, 1 \notin u \end{cases}.$$

Proof. By definition (see Theorem 5.1.1) we have

$$\widehat{a_0}(r, \infty) \equiv \begin{cases} (\perp, a_0 \blacktriangleright r), & r \geq 0 \\ (\top, \top), & r < 0 \end{cases} = \begin{cases} (\perp, [a_0 \ominus r]), & r \geq 0 \\ (\top, \top), & r < 0 \end{cases},$$

This comes to $(\perp, [a_0])$ when $r = 0$, and for $r > 0$ we get

$$\begin{aligned} \widehat{a_0}(r, \infty) &= (\perp, a_0 \blacktriangleright r) = (\perp, [a_0 \ominus r]) = \left(\perp, \left[r \left(\frac{a_0}{r} \ominus 1 \right) \right] \right) = \\ &= \left(\perp, \left[r \left(\frac{a_0}{r} - a_0 \right) \right] \right) = (\perp, [(1 - r)^+ a_0]) = \begin{cases} (\perp, [(1 - r)^+ a_0]) & , \quad 0 < r < 1 \\ (\perp, \perp) & , \quad r \geq 1 \end{cases} = \\ &= \begin{cases} (\perp, [a_0]) & , \quad 0 < r < 1 \\ (\perp, \perp) & , \quad r \geq 1 \end{cases}. \end{aligned}$$

The last equality results from the fact that truncation kernels are closed under scalar multiplication.

We claim that $[a_0] = \top$. This follows directly from two facts: first, $[a_0] \supseteq \overline{A}$ since a_0 is the greatest element of \overline{A} , and second, $[a_0]$ satisfies property (2) of Lemma 2.1.2. Thus we can summarize the situation as follows.

$$\widehat{a_0}(r, \infty) = \begin{cases} (\top, \top) & , \quad r < 0 \\ (\perp, \top) & , \quad 0 \leq r < 1 \\ (\perp, \perp) & , \quad r \geq 1 \end{cases}$$

In light of the fact that $\widehat{a}_0(-\infty, r) = \bigvee_{s < r} \widehat{a}_0(s, \infty)^*$, this information supports the inference that

$$\widehat{a}_0(-\infty, r) = \begin{cases} (\top, \top) & , \quad r > 1 \\ (\top, \perp) & , \quad 0 < r \leq 1 \\ (\perp, \perp) & , \quad r \leq 0 \end{cases}.$$

The proposition itself is a consequence of the last two displayed equations. \square

Let A be a trunc with spectrum $\mathcal{MA} = 2_F\mathcal{KA}$, and let ν_M designate the insertion $2_F\mathcal{KA} \rightarrow 2\mathcal{KA}$. Applying the $\mathcal{R}_{\mathbf{pF}}$ functor to ν_M and composing the result with μ_A provides a trunc injection $A \rightarrow B \equiv \mathcal{R}_{\mathbf{pF}}(2\mathcal{KA})$ which we denote by

$$\omega_A \equiv \mathcal{R}_{\mathbf{pF}}\nu_M \circ \mu_A.$$

We abuse the notation to the extent of using \widehat{a} to denote $\omega_A(a)$ and \widehat{A} to denote $\{\widehat{a} : a \in A\}$, trusting the reader to supply the appropriate meaning from context.

Now \mathcal{MB} , being isomorphic to $2\mathcal{KA}$, is an isolated point frame, so that \overline{B} has a greatest element b_0 by Proposition 6.1.1. We define

$$\omega_A \equiv \langle \widehat{A}, b_0 \rangle,$$

the subtrunc of B generated by $\widehat{A} \cup \{b_0\}$.

6.1.3. Theorem. *\mathbf{W} is monoreflective in \mathbf{AT} , and $\omega_A : A \rightarrow \omega_A$ is a reflector for $A \in \mathbf{AT}$.*

Proof. Consider the \mathbf{AT} -morphism $\theta : A \rightarrow C \in \mathbf{W}$, so that $\mathcal{MC} = 2\mathcal{KC}$ is isolated. Let $g : 2_F\mathcal{KA} \rightarrow 2\mathcal{KC}$ be the unique pointed frame map such that $\mathcal{R}_{\mathbf{pF}}g \circ \mu_A = \mu_C \circ \theta$. Now g extends uniquely over ν_M since the latter is the free isolated point frame over $2_F\mathcal{KA}$, thus providing a unique pointed frame map $g' : 2\mathcal{KA} \rightarrow 2\mathcal{KC}$ such that $g' \circ \nu_M = g$. Then, as the reader may easily check, $\mathcal{R}_{\mathbf{pF}}g' : \mathcal{R}_0(2\mathcal{KA}) \rightarrow \mathcal{R}_0(2\mathcal{KB})$, \square

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